ARITHMETIC

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FOR

SCHOOLS

BY

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PREFACE.

In the following work it has been the endeavor to put the science of Arithmetic on a sound basis, and to give clear and complete explanations of all the fundamental principles and processes. It has not been the aim to introduce novelties, but to promote accuracy and clearness of conception, so as to make the study of Arithmetic not only of practical utility, but also of great educational value.

I am indebted to many friends for their kindness in looking over the proof sheets, for help in the verification of the answers, and for valuable criticisms and suggestions. My special thanks are due to Mr. J. Barnard, M.A., Head Mathematical Master at Christ's Hospital.

CHARLES SMITH.

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ARITHMETIC.

CHAPTER I.

NUMERATION — NOTATION.

1. THE idea of **number** is first acquired from the observation of groups of distinct objects, actions, sounds, etc.; we thus learn to speak of *two* boys, *three* balls, *four* strokes of a clock, etc.

A single object of any kind, or any group of objects considered as a whole, is called a **unit.**

Thus, one *ball*, one *inch*, one *dozen*, *one*, one *ten*, are units.

2. Arithmetic is the science which treats of numbers and of the different operations to which they are subject.

3. The first few numbers in order are, *one, two, three, four, five, six, seven, eight, nine,* and *ten.*

4. It will be observed that the names of the first ten numbers are in no way connected with one another.

Now it is obvious that the knowledge of numbers and of their relations to one another must always have remained very limited if every successive number had had a special name given to it independent of the names of the preceding numbers; for it would be almost impossible to remember, in their order, many such names.

5. Successive numbers have therefore been named according to a systematic plan which requires the use of as few independent names as possible.

The method by which numbers are expressed in words according to some systematic plan is called **Numeration.**

NUMERATION.

6. To show how all numbers can be named by means of a few special words, imagine a collection of objects of the same kind, for example, a heap of apples; and suppose that we wish to know how many apples there are, and to give a name to this number.

If there are not more than ten apples altogether, we find the number at once by **counting** them, that is by saying in order the names one, two, three, etc., each time separating one of the apples from the original heap; and the name which is said with the last of the heap gives the number of the apples.

If there are more than ten apples in the heap, count off ten and put them apart, and go on making groups of ten until there are fewer than ten apples left. Suppose there are seven groups of ten each and five apples over, then we could call the number seven tens and five.

By separating the whole heap into groups of ten in this way we at once find, and can give a name to, the number of the apples, provided there are not more than ten of the groups. Thus, our original ten names suffice to name all numbers up to that which is made up of ten groups, each containing ten apples, and we have a new name, namely one *hundred*, for the number which consists of ten tens.

If there are more than ten of the groups each of which contains ten apples, the groups can be arranged in sets of

ten, so that there will be one hundred apples in each of these sets. Suppose that there are five of these sets and six groups over and four single apples besides, then the number is made up of five hundreds, six tens, and four. Thus no new name is necessary until we come to the number which consists of ten hundreds, and this number is called a *thousand*.

It will be seen at once that the names in actual use are only slightly modified forms of the names which naturally arise from the above method of division into groups of ten. Instead of saying two tens, three tens, four tens, etc., we say twenty, thirty, forty, etc., and we say seventy-five instead of seventy *and* five. Also instead of the names ten and one, ten and two, ten and three, ten and four, etc., we use the names *eleven* [Gothic *ainlif*, *ain* one and *lif* ten], *twelve* [Gothic *twalif*, *twa* two and *lif* ten], *thirteen*, *fourteen*, etc.

If an apple be cut into ten equal parts, any number of these parts may be put with some apples already counted. Each part is but one out of ten parts and may be counted as one *tenth*.

If one of these tenths be cut into ten equal parts, each new part is but one out of ten parts of one tenth, and may be counted as one *one-hundredth*. Thus by separating into ten equal parts, etc., we do not really require new names. If, now, we have five sets, six groups, four single apples, and three tenths and seven hundredths, then the number is five hundred sixty-four and thirty-seven hundredths (for three tenths is equal to thirty hundredths).

The principle of the ordinary system of numeration will now be apparent.

7. The English names which are employed in the system of Numeration which is universally used are the following: *one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seven-*

teen, eighteen, nineteen, twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety,

a *hundred*	which is	ten tens,
a *thousand*	" "	ten hundreds,
a *million*	" "	a thousand thousands,
a *billion*	" "	a thousand millions,
a *trillion*	" "	a thousand billions,
a *quadrillion*	" "	a thousand trillions,

and so on.

The names *billion, trillion,* etc., are very rarely used.

8. The numbers *one, ten, a hundred, a thousand, ten thousand, a hundred thousand, a million,* etc., are often called *units* of the *first order,* of the *second order,* of the *third order,* etc.; and ten units of any order are required to make one unit of the next higher order.

To give a name to any number it is sufficient to state the number of units of each different order that the number contains.

Thus the number which is made up of three *millions* two *hundreds-of-thousands* three *tens-of-thousands* four *thousands* five *hundreds* seven *tens* six *units* two *tenths* and four *hundredths* is called *three million two hundred thirty-four thousand five hundred seventy-six* and twenty-four *hundredths;* also the number which is made up of two *hundreds-of-millions* three *tens-of-millions* two *tens-of-thousands* three *thousands* and five *hundreds* is called *two hundred thirty million twenty-three thousand five hundred.*

It should be remarked that the parts of a number are mentioned in the order of their magnitude, the largest being given first; in English, however, this order is reversed for the numbers between ten and twenty. It should also be noticed that all the thousands, all the millions, all the billions, etc., are grouped together, as in the above two cases.

9. The system of numeration described above is called the **Decimal** system, since *ten* units of any order are required to make up one unit of the next higher order.

The decimal system of numeration is employed by all people who have any names at all for numbers greater than ten, and the origin of the system was doubtless the natural habit of counting on the fingers.

NOTATION.

10. We have now to show how numbers can be represented in a simple manner by means of a few symbols called *figures* or *digits.*

The method by which numbers are expressed by means of symbols according to some systematic plan is called **Notation.**

11. The first nine numbers in order are represented by the symbols

1, 2, 3, 4, 5, 6, 7, 8, 9.

The same figures are also employed to represent the first nine collections of *tens,* of *hundreds,* of *thousands,* etc., but on the understanding that the figures are to be written in a row, and that the figure which represents the units of the highest order named in a number is to be written as the left-hand figure of the row; while the figure which represents the units of the lowest order named is to be written as the right-hand figure. Thus, *any figure placed just to the right of another represents units of the order next below that represented by the other.*

To distinguish the figures which represent *units, tens, hundreds,* etc., from those which represent *tenths, hundredths,* etc., a dot, called the **Decimal Point,** must be

written between the figures which represent *units* and *tenths*.

Forty-five is written	45.
Four hundred seventy-two is written	472.
Three and two-tenths is written	3.2
Fifty-one and twenty-seven hundredths is written	51.27
Thirty-five hundredths is written	.35

12. The decimal point serves to separate the figures which represent the wholes from those which represent the tenths, hundredths, etc. That part of the number to the left of the decimal point is called the **Integral** part; that part to the right is called the **Decimal** part.

In 2.5, the 2 is integral, and the .5 is decimal; in 45.627, the 45 is integral, and the .627 is decimal.

The decimal point is omitted, if the number contains no decimal.

13. Besides the nine symbols already specified, it is necessary to have an additional symbol to meet the case when units of one or more of the different orders are absent. This symbol is 0; its name is *naught* or *cipher.* It has no value by itself, and is used to indicate that there are no units of the particular order corresponding to the place in which it occurs.

The other figures are sometimes distinguished from the naught by being called *significant figures.*

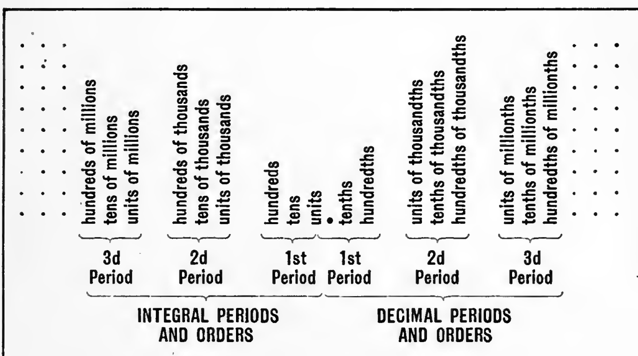
Thus, 20 represents two *tens* and no *ones;* that is, the number *twenty.* Again, 2005 represents two *thousands*, no *hundreds*, no *tens*, and five *ones;* that is, the number *two thousand five,* the naughts serving to bring the significant figures into the places intended for them.

It should be noticed that a naught placed at the *beginning* of an integral number, or at the *end* of a decimal number, does not affect the value of the number; 056, .7020, 0102.60 are the same as 56, .702, 102.6.

14. Any figure, say 5, has *two* values; namely (1) its **digit value**, which is indicated by its shape, and in virtue of which it always represents *five* units of some kind, and (2) its **local value**, which depends only on its place, and in virtue of which it represents units of the order which corresponds to its position in the horizontal row; so that if 5 be in the first place to the left of the decimal point, it represents five units; if in the second place, it represents five tens, and so on; if it be in the first place to the right of the decimal point, it represents 5 tenths, and so on.

Thus, in 45 the 5 represents five *ones;* in 567, it represents five *hundreds;* in 2.35, it represents five *hundredths.* Again, 534.62 represents five *hundreds* three *tens* four *ones* six *tenths* and two *hundredths;* therefore the number represented is *five hundred thirty-four* and *sixty-two hundredths.*

15. The names of the units of the orders in common use, and the positions which correspond to them, are shown below:



It will be seen that the names are repeated in groups of three; and that the decimal periods and orders correspond in name to the integral periods and orders.

The first twelve integral periods are as follows:

First,	Units.	*Seventh,*	Quintillions.
Second,	Thousands.	*Eighth,*	Sextillions.
Third,	Millions.	*Ninth,*	Septillions.
Fourth,	Billions.	*Tenth,*	Octillions.
Fifth,	Trillions.	*Eleventh,*	Nonillions.
Sixth,	Quadrillions.	*Twelfth,*	Decillions.

16. To write in figures any number expressed in words, it is necessary only to write the figures which represent the number of the units of the different orders in their proper places as shown above, filling up the vacant places, if any, with naughts.

Thus, to write in figures the number two hundred forty-three, we must have 2 in the place for *hundreds*, 4 in the place for *tens*, and 3 in the place for *units*. To write in figures the number five hundred twenty-four thousand six hundred seven, we must have 5 in the place for *hundreds of thousands*, 2 in the place for *tens of thousands*, 4 in the place for *units of thousands*, 6 in the place for *hundreds*, 0 in the place for *tens*, and 7 in the place for *units*, as follows: 524,607. Sixteen million is written 16,000,000. To write in figures one thousand thirty and seven thousandths, we must have 1 in *thousands*' place, 0 in *hundreds*' place, 3 in *tens*' place, 0 in *units*' place, 0 in *tenths*' place, 0 in *hundredths*' place, and 7 in *thousandths*' place, as follows: 1030.007.

17. To express in words any number given in figures, first divide the integral and the decimal parts separately into groups of three, beginning at the right in each case (the left-hand groups will often be incomplete); beginning at the left, read each group of the *integral* part as if it were alone and give it the name of the period to which it belongs, then read the *decimal* part as if it were integral and give it the name of the order on the right.

For example, to express in words the number represented by 24160523, we can separate off two complete groups of three figures, and 24,160,523 is then read twenty-four million one hundred sixty thousand five hundred twenty-three.

To express in words the number represented by 36405.4916, we can separate off one complete group in the integral part and one in the decimal part, and 36,405.4,916 is then read thirty-six thousand four hundred five and four thousand nine hundred sixteen ten-thousandths.

To '**read off**' decimals, it is, however, the common practice merely to name the digits in order.

Thus .615 is read 'decimal six, one, five'; 15.0524 is read 'fifteen, decimal naught, five, two, four'; and 1567.0082 is read 'one thousand five hundred sixty-seven, decimal naught, naught, eight, two.'

In reading numbers the word 'and' should be used *only* when we reach a decimal point.

EXAMPLES I.

1. For what does 5 stand in the numbers 15, 1.57, 514, 352167, and 3561234, respectively?

2. For what does 7 stand in the numbers 70, 37123, 125.479, 274126315, and 370001002003, respectively?

3. Name all the figures which represent their digit value of *hundreds* in 314, 2167, 50412, and 31024.

4. Name all the figures which represent their digit value of *thousands* in 2314, 56123, 60417, and 3005167.

5. Express in words the separate value of every figure in 3.5, 15.7, 125.34, 12.53, 800.17, 1200.63, .875, 50.037, 5.00107, 560002.19007.

6. Express in words the numbers 27, 349, 560, 3.06, 1204, and 5020.

7. Express in words the numbers 200.9, 6050, 12345, 10.305, 40050, and 1.20463.

8. Express in words the numbers 518618, 602010, 100010, 504075, 420040, and 107.005.

9. Express in words 111111111, 1203405, 2314100, 504.0314, 20050060, and 30300074.

10. Express in words the numbers 3012004, 1101.11011, 201201201, 1000040305101, and 604102000300004.

11. Write in figures the numbers fifty-eight, eighty-five, two hundred eleven, three thousand twelve, six thousand forty, and nine thousand three hundred.

Write in figures the following numbers:

12. Twelve and three-tenths.

13. Three hundred four and nine-tenths.

14. Twenty-five, three-tenths, and four hundredths.

15. Four, six-tenths, and seven-thousandths.

16. One million, four-tenths, and three-millionths.

17. Write in figures the numbers eleven hundred eleven, fourteen hundred sixty, twelve hundred thousand sixteen hundred, six million twelve hundred sixteen, and eleven billion eleven hundred eleven.

18. Write in figures twenty million twenty thousand, seventeen million fifty thousand nineteen, one hundred four million six hundred two thousand eleven, and six thousand three hundred seven million two thousand fifty six.

18. The ordinary system of notation was introduced into Europe by the Arabians, and is still called the **Arabic system** of Notation although it is now known that the Arabians derived their knowledge from the Hindoos.

ROMAN NUMERALS.

19. Besides the Arabic system of notation some use is still made of the cumbrous system employed by the Romans.

The symbols which were used by the Romans, and which are called **Roman Numerals**, are the following:

I for 1, V for 5, X for 10, L for 50, C for 100, D for 500, M for 1000.

A horizontal line over any numeral increases its value one thousand fold: thus $\overline{\text{V}}$ stands for 5000, $\overline{\text{X}}$ for 10000, etc.

Roman numerals are arranged in order of magnitude from left to right, and are repeated as often as may be necessary.

Thus, 2 is represented by II, 30 by XXX, 233 by CCXXXIII, and 1887 by MDCCCLXXXVII.

20. To avoid some of the troublesome repetitions which are common to the Roman system of notation, a numeral is in certain cases placed *before* another of greater value to denote that the value of the larger is to be *diminished* by the amount of the smaller.

Thus, IV denotes one less than five, that is, 4; IX denotes one less than ten, that is, 9; XL denotes ten less than fifty, that is, 40; and XC denotes ten less than one hundred, that is, 90; so also, CCXC denotes 290.

21. The symbols CIↃ, CCIↃↃ, CCCIↃↃↃ, etc., were anciently employed to denote respectively 1000, 10,000, 100,000 etc.; also IↃ, IↃↃ, IↃↃↃ, etc., to denote respectively 500, 5000, 50,000, etc. In fact, M and D are only modified forms of CIↃ and IↃ respectively.

22. Roman numerals were used only to *register* numbers, and were never employed in making numerical calculations. The Romans made their calculations by means of counters and a mechanical apparatus called an *Abacus.* The counters used were often pebbles (Latin, *calculus*), whence our word *calculation.*

EXAMPLES II.

1. Express all the numbers from 1 to 20 by means of Roman numerals.

2. Express by means of Roman numerals the numbers 20, 30, 40, 50, 60, 70, 80, 90, 200, 400, 600, 800, and 900.

3. Express by means of Roman numerals the numbers 39, 49, 59, 69, 79, 89, 99, 96, 444, 1294, and 1889.

4. Write the numbers LVIII, XXXIX, XLIV, XCIV, XCIX, CXCIX, and MMDCCXCIX, in the Arabic notation.

CHAPTER II.

ADDITION — SUBTRACTION — MULTIPLICATION — DIVISION.

ADDITION.

23. THE process of finding a single number which contains as many units as there are in two or more given numbers *taken together* is called **Addition**; and this single number is called the **Sum**.

The sum of the numbers of the units in two or more groups would therefore be found by forming a single group containing them all, and then counting the number of the units in this single group.

24. The following fundamental truth is evident:

The number of the things in any group will always be found to be the same in whatever order they may be counted.

From this it follows that the sum of the numbers of the things in any two groups will be found by first counting all the things in the first group and then proceeding to the second; that is, by *increasing the number in the first group by as many units as there are in the second.* The same sum will also be found *by increasing the number in the second group by as many units as there are in the first.*

Thus, the sum of 3 and 5 is found by counting five onwards from three, namely four, five, six, seven, *eight;* or by counting three onwards from five, namely six, seven,

eight. In the first case we are said to add 5 to 3, and in the second case we are said to add 3 to 5; but the results must be the same.

25. Addition is indicated by the sign $+$, which is read '**plus.**'

Thus, $5 + 4$ is read five plus four, and denotes that 5 is to be *increased by* 4, that is, that 4 is to be added to 5; also, $5 + 4 + 3$ denotes that 4 is to be added to 5, and then 3 added to the result.

26. The sign $=$, which is read '*equals*' or '*is equal to,*' is used to denote the equality of two numbers.

Thus, $5 + 4 = 9$ denotes that the sum of 5 and 4 is 9.

27. When children first begin to add they make use of their fingers, but all counting on the fingers, or with any other real objects, should be discontinued as soon as possible, and the results of adding numbers not greater than nine should be given instantaneously.

Tables of the results of the addition of any two numbers each not greater than 10 might at first be made by the pupil, arranged in lines; as for example, 8 and 1 are 9, 8 and 2 are 10, 8 and 3 are 11, etc.

EXAMPLES III.

Oral Exercises.

These examples should be practised until great rapidity is attained.

1. Add 1 and 9, 3 and 8, 2 and 6, 4 and 7, 6 and 3, 4 and 4.

2. Add 7 and 8, 7 and 6, 3 and 9, 5 and 4, 3 and 5, 9 and 8.

3. Add 4 and 3, 9 and 9, 8 and 8, 6 and 9, 7 and 2, 3 and 3.

4. Add 5 and 9, 9 and 4, 6 and 8, 5 and 7, 2 and 9, 8 and 5.

5. Add 7 and 7, 5 and 5, 6 and 6, 8 and 4, 6 and 4, 9 and 7.

6. Add 8 to 15, to 25, to 35, to 45, to 65, and to 95.

7. Add 13 and 7, 23 and 7, 43 and 7, 63 and 7, 83 and 7, 93 and 7.

8. Add 9 to 17, to 27, to 57, to 67, to 87, and to 97.

9. Begin with 7 and add 2 again and again up to 27.

Do not say 7 and 2 are 9 and 2 are 11 and 2 are 13, etc., but state results; thus, 7, 9, 11, 13, etc.

10. Begin with 2 and add 3 again and again up to 35.

11. Begin with 85 and add 4 again and again up to 101.

12. Begin with 50 and keep on adding sevens until the sum exceeds 100.

13. Begin with 15 and keep on adding nines until the sum exceeds 100.

14. Add the following numbers in order, first beginning at the right and then at the left:

(1) 2, 7, 4, 0, 6, 9, 5, 2, 6, 5, 9, 3, 4, 8.

State results only; thus, 2, 9, 13, 13, 19, 28, etc.

(2) 7, 9, 5, 4, 0, 8, 6, 7, 3, 5, 9, 8, 2, 6.

(3) 3, 5, 6, 9, 0, 7, 8, 4, 3, 6, 2, 5, 7, 9.

(4) 9, 6, 7, 4, 2, 8, 1, 3, 7, 5, 4, 6, 5, 8.

28. The sum of *any* two numbers may be found by counting onwards from the first as many units as there are in the second, but this method would obviously be very troublesome except when the second number is very small.

Now numbers are divided, as we have already learned, into groups of units, tens, hundreds, tenths, hundredths, etc.; and when numbers are to be added, the parts into which they are divided may be added in *any order* we please, provided they are all counted; hence *we may first add the units of one order, then the units of another order, and so on.*

29. In order to add numbers, they should first be arranged so that *their decimal points are in a vertical column.* This will ensure that all the tenths shall be in the same vertical column, and so for the hundredths, etc.; and so also for the units, tens, hundreds, etc. This arrangement is for convenience only.

The following examples will show how this principle enables us readily to find the sum of any given numbers.

Ex. 1. *Add* 235.7 *and* 524.2.

Since we wish to add the tenths by *them*selves, the units by *them*selves, etc., we write the numbers so that the decimal points are in a vertical column; thus,

$$\begin{array}{r} 235.7 \\ 524.2 \end{array}$$

Now 2 tenths and 7 tenths make 9 tenths, 4 units and 5 units make 9 units, 2 tens and 3 tens make 5 tens, and 5 hundreds and 2 hundreds make 7 hundreds. The required sum is generally placed just under the numbers to be added and separated from them by a horizontal line; thus,

$$\begin{array}{r} 235.7 \\ 524.2 \\ \hline 759.9 \end{array}$$

Ex. 2. *Add* 548.6, 789, *and* 197.8.

Write the numbers as in Ex. 1; thus,

$$\begin{array}{r@{}l} 548&.6 \\ 789&. \\ 197&.8 \\ \hline 1535&.4 \end{array}$$

Now 8 tenths and 6 tenths make 14 tenths, that is, 1 unit and 4 tenths. The 4 tenths can be put in the column for tenths, but the 1 unit must be counted with the other units. We then have 1 unit, 7 units, 9 units, and 8 units, which make 25 units; that is, 2 tens and 5 units. The 5 is put in the column for units, but the 2 tens are 'carried' (as it is called) and added with the other tens. So we proceed until all the columns are added.

NOTE. Since ten units of *any* order make one unit of the next higher order, the figures in any column may be added without specifying the kind of units they represent; that is, without calling them *tens*, or *hundreds*, or *thousands*, etc., as the case may be.

Also, we should *never* use as many words as in the above explanations, but should say (see ex. 2) only 8, 14; 1 (carried), 8, 17, 25; 2 (carried), 11, 19, 23; 2 (carried), 3, 10, 15. Of course the 4, 5, 3, and 15 are the figures to be written. In all cases the *sums* of numbers should be more prominent than the numbers themselves.

30. To detect mistakes in addition it is well to add each line of figures *twice*, once from bottom to top and once from top to bottom. An error is much more likely to be detected in this way than by simply repeating the addition in the same order, for the same mistake is very likely to be made again.

Pupils should *not be allowed* to add more than one column at a time.

EXAMPLES IV.

Written Exercises.

1. Add 3104, 297, 5649, and 989.

Find the sum of

2. 21.63, 5.24, 170.63, 27.59, 17.
3. 301.7, 30.17, 3.017, .3017, .03017.
4. 319, 562, 1230, 857, 4908, and 9087.
5. 235, 796, 804, 987, 359, and 856.
6. 170.2, 3.605, 17.35, 15.609, .0086.

7. .0037, 21.85, 169.4, 17.9375, .90087.

8. 4.1372, 41.372, 4137.2, .41372, 41372.

Add the numbers in each column and in each row of the squares. Do not change the positions of the numbers.

9.

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

10.

9	16	23	12	5
13	2	10	19	21
20	24	11	3	7
1	8	17	25	14
22	15	4	6	18

11.

1	101	80	59	38	17	117	96	75	54	33
21	121	89	68	47	26	5	105	84	63	42
30	9	109	88	56	35	14	114	93	72	51
39	18	118	97	76	55	23	2	102	81	60
48	27	6	106	85	64	43	22	111	90	69
57	36	15	115	94	73	52	31	10	110	78
77	45	24	3	103	82	61	40	19	119	98
86	65	44	12	112	91	70	49	28	7	107
95	74	53	32	11	100	79	58	37	16	116
104	83	62	41	20	120	99	67	46	25	4
113	92	71	50	29	8	108	87	66	34	13

Perform the additions indicated below:

12.

3157
294
16903
8057
62934
998

13.

589.761
35.71
840.693
392.75
1569.4242
359.177

14.

412.64506
39.17412
246.82441
49.1733
387.198207
129.38946

15.	16.	17.
50971	314569	842713
8265	73985	9185
13926	387648	38977
78912	930807	796359
34056	186794	246824
19389	389548	135791
8747	153875	924678

In the next three examples do not change the positions of the numbers.

18. Find $30.1 + 297 + 35.16 + 1079 + 8.017 + 10.053$.

19. Find $93084 + 15614 + 3801.76 + 536174 + 123456 + 40.404$.

20. Find $218904 + 37.215 + .199 + 582163 + 397157 + 81.429 + 7.9163$.

21. Add six hundred ninety-five, one thousand seventy-four, eleven thousand four hundred eighty-nine, and fifty-four thousand three hundred seventy.

22. Add three million four hundred seventeen thousand thirty-five, nine hundred forty-six thousand seven hundred, fifteen million fifteen thousand fifteen, and sixty million sixteen hundred twenty-four.

23. Add six million five hundred nine thousand seven hundred six and twelve thousand four hundred thirty-two hundred-thousandths, three hundred ninety thousand and four hundred twelve thousandths, eighteen million forty and six ten-thousandths.

31. Thus far we have studied numbers without reference to objects.

When numbers are used without reference to any particular units, they are called **Abstract Numbers.**

Two and *five* are abstract numbers.

When numbers are associated with particular units, they are called **Concrete Numbers.**

Two feet and *five tons* are concrete numbers.

32. Concrete numbers can be added only when *the unit is the same.* For example, 3 horses and 4 cows do not make 7 horses nor 7 cows; they do, however, make 7 *animals;* because regarding them as animals the unit is the same. Also the sum of 3 feet and 4 inches is not 7 feet nor 7 inches.

EXAMPLES V.

Written Exercises.

1. In 1890 the population of each of the New England States was as follows: Maine, 661000; New Hampshire, 377000; Vermont, 332000; Massachusetts, 2239000; Rhode Island, 346000; Connecticut, 746000. What was the total?

2. In a town, noted for the number of its schools, there were 225 boys in a military school, 175 girls in a school for girls, 126 young men in a theological school, 163 boys in a training school, 23 children in a kindergarten, and 1500 pupils in the public schools. How many pupils in all?

3. A man paid 527.37 dollars for 14 cows, 1463.80 dollars for twelve horses, and 918.36 dollars for 153 pigs. How many animals were there, and how much was paid for them all?

4. The population of each of the six northern counties of England is as follows: Cumberland, 250647; Durham, 867258; Lancashire, 3454441; Northumberland, 434086; Westmoreland, 64191; and Yorkshire, 2886564. What is the total population?

SUBTRACTION.

33. The process of finding how many units are left when a number is *taken away* from a larger number is called **Subtraction.** The result is called the **Remainder,** or the **Difference.**

Any two numbers can be added; it is, however, impossible to subtract one number from a smaller number.

34. The larger of the two numbers is called the **Minuend.**

The smaller of the two numbers is called the **Subtrahend.**

ILLUSTRATION.

8	Minuend.
5	Subtrahend.
3	Remainder.

35. It is clear that *the remainder is that number which, when added to the subtrahend, will give the minuend.*

Thus, to subtract 5 from 12 is to find the number which, when added to 5, will make 12.

The question involved in subtraction may be put in different ways. Thus, it may be asked:

(1) What is the remainder when 5 is taken from 12?

(2) What must be added to 5 to make 12?

(3) By how many is 12 greater than 5?

(4) By how many is 5 less than 12?

36. Subtraction is indicated by the sign $-$, which is read '**minus.**'

Thus, $9 - 4$ is read nine minus four, and denotes that 9 is to be diminished by 4, that is, that 4 is to be subtracted from 9; also, $5 - 4 + 3$ denotes that 4 is to be taken from 5, and then 3 added to the result.

37. The knowledge of the results of the addition of numbers not greater than ten will furnish us with the

results of the subtraction of small numbers. Examples of subtractions of this kind should be practised until great rapidity is attained.

EXAMPLES VI.

Oral Exercises.

1. How many are left when we take 7 from 14, 5 from 10, 6 from 12, 8 from 12, 4 from 10, and 7 from 16, respectively?

2. How many are left when we take 5 from 14, 4 from 13, 8 from 14, 7 from 12, 9 from 11, and 5 from 13, respectively?

Find the difference between the numbers in each of the following pairs:

3. 5 and 12, 7 and 16, 9 and 18, 3 and 11, 6 and 14, 8 and 15.

4. 3 and 8, 5 and 11, 6 and 13, 8 and 14, 7 and 15, 9 and 16.

5. Begin with 50 and go on diminishing by fours as many times as possible.

6. Begin with 53 and go on diminishing by fives as many times as possible.

7. Begin with 70 and go on diminishing by sixes as many times as possible.

8. What must be added to 5 to make 8, to make 13, to make 10, to make 12?

9. What must be added to 7 to make 9, to make 12, to make 10, to make 15?

10. What must be added to 8 to make 10, to make 12, to make 14, to make 16?

Fill up the blanks below.

11.	9 and	make 10, 3 and	make 11, 2 and	make 8,
	4 and	make 11, 6 and	make 9, 4 and	make 8.
12.	7 and	make 15, 6 and	make 13, 9 and	make 12,
	4 and	make 9, 3 and	make 8, 8 and	make 17.
13.	3 and	make 7, 9 and	make 18, 8 and	make 16,
	6 and	make 15, 7 and	make 9, 3 and	make 6.

38. The consideration of the following examples will show how the difference between any two numbers can be found.

Ex. 1. *Subtract* 524.63 *from* 759.85.

The smaller number should be placed just under the greater, so that one decimal point is vertically over the other. (See Art. 29.)

$$\begin{array}{r} 759.85 \\ \underline{524.63} \end{array}$$

Beginning with the lowest order, we find the remainder when 3 hundredths are taken from 5 hundredths, 6 tenths from 8 tenths, 4 units from 9 units, 2 tens from 5 tens, and 5 hundreds from 7 hundreds; thus,

$$\begin{array}{rl} 759.85 & \text{Minuend.} \\ \underline{524.63} & \text{Subtrahend.} \\ 235.22 & \text{Remainder.} \end{array}$$

Ex. 2. *Subtract* 35.7 *from* 78.3.

$$\begin{array}{rl} 78.3 & \text{Minuend.} \\ \underline{35.7} & \text{Subtrahend.} \\ 42.6 & \text{Remainder} \end{array}$$

Now 7 tenths are more than 3 tenths, therefore we cannot subtract: if, however, we take 1 unit from the 8 units and change that unit to 10 tenths, we shall have 13 tenths in all. Now 7 tenths from 13 tenths leave 6 tenths, 5 units from 7 units leave 2 units, and 3 tens from 7 tens leave 4 tens. Remainder = 42.6.

Mental Work Illustrated. We may omit names of orders. (See note, Art. 29.)

Ex. 3.	468.27	9 from 17, 8.
	186.49	4 from 11, 7.
	281.78	6 from 7, 1.
		8 from 16, 8.
		1 from 3, 2.
Ex. 4.	20.07	0 from 7, 7.
	12.6	6 from 10, 4.
	7.47	2 from 9, 7.
		1 from 1, 0.

In this example 1 ten is taken from 2 tens and changed to 10 units; one of these units is changed to ten tenths. The operation may be represented thus:

20.07	= 1	9.	10	7
12.6	= 1	2.	6	
Remainder =		7.	4	7

39. One concrete number cannot be subtracted from another unless both are expressed in terms of the same unit. For example, we cannot subtract 5 tons from 7 miles; nor can we subtract 3 feet from 60 inches, unless either 3 feet is expressed in inches or 60 inches expressed in feet.

40. It is easily seen that if from a given number several numbers be taken in succession the result will be the same as if the *sum* of those numbers were subtracted from the given number.

Ex. *Subtract the sum of* 366, 648, *and* 759 *from* 2314.

$$\begin{array}{r} 2314 \\ \hline 366 \\ 648 \\ 759 \\ \hline 541 \end{array}$$

9, 8, and 6 make 23; subtract the 3 from the 4 and carry the 2; 2, 5, 4, and 6 make 17; subtract the 7 from 11 and carry the 1; 1, 7, 6, and 3 make 17, which is to be subtracted from 22.

MENTAL WORK.

9, 17, 23,	3 from 4 = 1.
2, 7, 11, 17,	7 " 11 = 4.
1, 8, 14, 17,	17 " 22 = 5.

41. When several operations of addition and subtraction have to be performed in succession the result is the same *in whatever order the operations are performed.*

Hence, to find $28 - 15 + 26 - 17 - 14 + 12$, first find the sum of 28, 26, and 12, the numbers to be added; then the sum of 15, 17, and 14, the numbers to be subtracted; and finally taking the difference of these two sums; thus,

$$\begin{array}{r@{\quad}r} 28 & 15 \\ 26 & 17 \\ \underline{12} & \underline{14} \\ 66 - & 46 = 20. \end{array}$$

42. To detect mistakes in subtraction, add the remainder to the subtrahend, and the sum should equal the minuend; or subtract the remainder from the minuend, and the new remainder should equal the subtrahend.

EXAMPLES VII.

Written Exercises.

1. Subtract 129.6 from 3145, 81.7 from 3002, and 123.4 from 432.1.

2. Subtract 15.97 from 79.15, 18235 from 1000000, and 135.79 from 24680.6.

3. Find the values of $645 - 378$, $307 - 149$, $294 - 208$, $2179 - 1984$, $3206 - 1679$, and $120573 - 98765$.

Find the difference between

4. 3.726 and 5.949.
5. 14.753 and 6.876.
6. 1 and .888.
7. .00013 and .00175.
8. 3.008 and 3.08.
9. .217 and .271.
10. 20 and .675.
11. .8017 and .00693.

12. Find the values of

(1) $31 + 97 - 23 + 175 - 184$.

(2) $151 - 77 + 94 - 111$.

(3) $315 - 127 - 172 + 358 - 265$.

(4) $742 - 329 - 197 + 215$.

13. Find $3.17 + 4.216 - 5.8004 + 2.0097 - .99873$.

14. Find $21.09 - 3.985 - 7.0095 + .09372 - 4.38009 + 2.60009$.

15. Subtract from 11.214 the sum of 2.301, 1.7293, 2.0507, and 3.62743.

16. Subtract from 20 the sum of 3.416, 2.6008, 5.73124, and 1.5063.

17. Subtract from 121097 the sum of 7916, 1214, 1397, and 34162.

18. Subtract from 1000000 the sum of 421654, 127, 31562, 1795, and 123456.

19. Subtract 27 from 80, and then 27 from the remainder, and so on as many times as possible; and find the final remainder.

20. What number must be taken from 81 to leave 37 as remainder?

21. By how much does the sum of 3.5612 and 4.71305 exceed the sum of 1.70862 and 5.91927?

22. What number must be taken from one hundred thousand to leave five thousand four hundred eighty-seven as remainder?

23. The difference between two numbers is 145, and the greater is 597; what is the smaller?

24. The sum of two numbers is 1000, and one of them is 594; what is the other?

25. On a man's birthday in 1891 he was 63 years old. In what year was he born?

26. In 1891 a man of 65 was on his birthday just 37 years older than his son. In what year was the son born?

27. Add the sum of 516 and 784 to the difference between 314 and 176.

28. Add the difference between 1925 and 1789 to the difference between 3421 and 1679.

29. In an orchard there are 1572 fruit trees; of these 352 are apple trees, 275 are pear trees, and 187 are plum trees. How many other trees are there?

30. The population of each of five towns is as follows: *A*, 3789; *B*, 7861; *C*, 2893; *D*, 756; *E*, 847. If *B* and *D* were united, the new town would be how much larger than *A*, *C*, and *E* together?

MULTIPLICATION.

43. A short process of *adding two or more equal numbers* is called **Multiplication.**

Ex. 1. $5 + 5 + 5 + 5 = 20$; *i.e.*, 4 fives $= 20$.
Ex. 2. $3 + 3 + 3 + 3 + 3 = 15$; *i.e.*, 5 threes $= 15$.

If we say (Ex. 1) 5, 10, 15, 20, or (Ex. 2) 3, 6, 9, 12, 15, we are *adding* by a long process.

If we say 4 fives $= 20$, or 5 threes $= 15$, we are *adding* by a short process called *multiplication.*

44. The number which is to be thus increased is called the **Multiplicand.**

The number which indicates how many equal numbers are to be added is called the **Multiplier.**

The result of multiplication is called the **Product.**

The multiplicand and multiplier are called **Factors** of the product.

Ex. 1. *Multiply* 5 *by* 4. Ex. 2. *Multiply* 3 *by* 5.

$$\text{Factors of } 20\left\{\begin{array}{r}5\\ \underline{4}\\ 20\end{array}\right. \quad \begin{array}{l}\text{Multiplicand.}\\ \text{Multiplier.}\\ \text{Product.}\end{array} \quad \left.\begin{array}{r}3\\ \underline{5}\\ 15\end{array}\right\}\text{Factors of } 15.$$

45. The multiplication of any two numbers not greater than nine is easily found by actual addition. It will be shown that every case of multiplication can be reduced to a series of cases of multiplications of numbers not greater than ten; it is therefore essential to learn by heart all the products of such numbers. These products are given in the following table, called the **Multiplication Table.**

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	27	84	96	108	120	132	144

Any horizontal line in the table gives the products of the number which begins the line by the first twelve numbers in order. Thus the fourth line can be read 1 four is 4, 2 fours are 8, 3 fours are 12, 4 fours are 16, etc.

It is usual and desirable, though not absolutely necessary, to learn the Multiplication Table as far as 12 times 12. This table

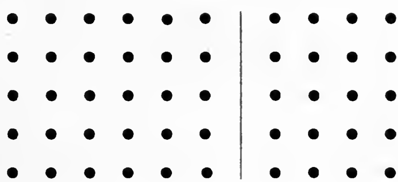
should be made again and again by the pupil himself by actual addition.

46. Multiplication is indicated by the sign $\times$, which is read **'multiplied by'** or **'times.'**

Thus, 5×4 is read 5 multiplied by 4, which means 4 times 5; also, $5 \times 4 \times 3$ denotes that 5 is to be multiplied by 4, and this product multiplied by 3.

When one number is multiplied by two or more other numbers *in succession*, the result is called the **Continued Product.**

47. Before considering how to find the product of any two numbers, certain general truths, which hold good for all numbers whatever, must be investigated. For this purpose consider the following arrangement of dots:



The total number of the dots is independent of the way in which they are counted.

Now there are 10 dots in each row and 5 rows; the whole number of the dots can therefore be counted as 10 repeated 5 times, or 5 repeated 10 times; *i.e.*, $10 \times 5 = 5 \times 10$. It is clear that this result would hold good however many rows and columns there might be; thus we are led to

Theorem I. *The product of any number by any second number is the same as the product of the second by the first.*

Again, if we consider separately the two parts divided by the vertical line, we see that the whole number of dots is the sum of 6 repeated 5 times and 4 repeated 5 times, the 6 and 4 together making 10, so that 10×5 is the same as $6 \times 5 + 4 \times 5$; thus we are led to

Theorem II. *The product of any two numbers is the same as the sum of the products of the multiplier and any two or more numbers which together make up the multiplicand.*

Now consider the following arrangement:

5	5	5	5	5	5
5	5	5	5	5	5
5	5	5	5	5	5

Here the sum of all the numbers consists of 5 repeated 6×3 times. But in each row 5 is repeated 6 times and the rows are repeated 3 times; hence the sum of all the numbers is $5 \times 6 \times 3$. Again, in each column 5 is repeated 3 times and the columns are repeated 6 times; hence the whole sum is $5 \times 3 \times 6$. Thus, to multiply by 6 and 3 in succession, in any order, gives the same result as to multiply at once by 6×3, that is by 18. It is clear that the same would be true for any other numbers whatever; thus we are led to

Theorem III. *To multiply by two or more numbers in succession gives the same result as to multiply at once by their product.*

48. To multiply by 10, 100, etc. Numbers are separated into groups of units, tens, hundreds, etc.; tenths, hundredths, etc.; and a number is multiplied by 10 when each of its parts is multiplied by 10 [Theorem II]; *i.e.*, when each of its parts is raised to the next higher order.

For example, 62.34 is multiplied by 10 when its 4 hundredths are made 4 tenths, its 3 tenths are made 3 units, its 2 units are

made 2 tens, and its 6 tens are made 6 hundreds; this is accomplished by moving the decimal point *one* place to the right; *i.e.*, $62.34 \times 10 = 623.4$.

Also, $623.4 \times 10 = 6234$; $6234 \times 10 = 62340$.

Multiplying by 10 and then by 10 again is the same as multiplying by 100, and it will be noticed that in multiplying 62.34 by 100, the decimal point is moved *two* places to the right.

Hence, *to multiply by* 10, 100, 1000, *etc., move the decimal point as many places to the right as there are naughts in the multiplier.*

EXAMPLES.
1. $164.2789 \times 10 = 1642.789$.
2. $164.2789 \times 1000 = 164278.9$.
3. $340 \times 100 = 34000$.

49. The following examples will show how to find the product of any two numbers.

Ex. 1. *Multiply* 52.34 *by* 7.

In multiplication, the multiplier is placed under the multiplicand, so that the right-hand figures shall be in the same vertical column. This is only for convenience.

52.34	Multiplicand.
7	Multiplier.
366.38	Product.

By Theorem II, we multiply the units of the different orders separately by 7, and add the results. Now, 4 hundredths $\times$ 7 = 28 hundredths, or 2 tenths and 8 hundredths; the 8 is put in hundredths' column, and the 2 must be counted with the tenths, or 'carried.' Next, 3 tenths $\times$ 7 = 21 tenths, which with the 2 tenths carried = 23 tenths, or 2 units and 3 tenths; write the 3 in tenths' column, and carry the 2 to units' column. Next, 2 units $\times$ 7 = 14 units, which with the 2 units carried = 16 units, or 1 ten and 6 units; write the 6 in units' column, and carry the one to tens' column. Finally, 5 tens $\times$ 7 = 35 tens, which with the 1 ten carried = 36 tens; write the 5 in tens' column, and the 3 in hundreds' column. The product is 366.38.

Ex. 2. *Multiply* .3 *by* .2.

$$\begin{array}{rl} .3 & \text{Multiplicand.} \\ \underline{.2} & \text{Multiplier.} \\ .06 & \text{Product.} \end{array}$$

Now, 3 tenths $\times$ 2 = 6 tenths. Since 2 is 10 times 2 tenths, the product obtained by multiplying by 2 is 10 times the product obtained by multiplying by 2 tenths. If, therefore, we separate the 6 tenths, which equals 60 hundredths, into 10 equal parts, one of those parts must be the true product. We find, from the multiplication table, that 6 hundredths is one of the 10 equal parts of 60 hundredths. Therefore .06 is the true product.

Ex. 3.

$$\begin{array}{l} .08 \times 3 = .24;\ .08 \times .3 = .024. \\ .18 \times 4 = .72;\ .18 \times .4 = .072. \\ .53 \times 2 = 1.06;\ .53 \times .2 = .106. \end{array}$$

Observe that the number of decimal places in the product is equal to the number of decimal places in both multiplicand and multiplier.

Ex. 4. *Multiply* 5.64 $\times$ 302.6.

$$\begin{array}{lrl} & 5.64 & \text{Multiplicand.} \\ & \underline{202.6} & \text{Multiplier.} \\ \text{1st partial product} & 3.384 & = 5.64 \times .6 \\ \text{2d partial product} & 11.28 & = 5.64 \times 2 \\ \text{3d partial product} & \underline{1692.} & = 5.64 \times 2 \times 100 \\ \text{Sum of partial products} & 1706.664 & = \text{Product.} \end{array}$$

We multiply 5.64 separately by .6, by 2, and by 200, writing the partial products in form for addition; *i.e.*, so that the decimal points are in column.

It is not necessary to specify, as we have done above, the kinds of units which are being multiplied at any stage. (See note, Art. 29.)

Ex. 5. *Multiply* 321 $\times$ 218.

$$\begin{array}{rl} 321 & \text{Multiplicand.} \\ \underline{218} & \text{Multiplier} \\ 2568 & = 321 \times 8 \\ 321\not{0} & = 321 \times 10 \\ \underline{642\not{0}\not{0}} & = 331 \times 200 \\ 69978 & = \text{Product.} \end{array}$$

The naughts are omitted in practice, because they count for nothing in addition.

Ex. 6.

```
       31642
      100506
      ------
      189852
    158210
  31642
  ----------
  3180210852
```

Ex. 7.

```
        27.006
      2001.908
      --------
        21 6048
     24 30 54
     27 00 6
  54012
  ------------
  54063.527448
```

Ex. 8.

```
  27.68
  36000
  -----
  16608
  8304
  ---------
  996480.00
```

Naughts at the right of multiplicand or multiplier are omitted in the partial products and annexed to the significant figures of the answer *before writing the decimal point.*

50. In all cases of multiplication **the multiplier must be an abstract number,** for to repeat anything 5 shillings times or 3 tons times is absurd; the multiplicand, however, may be either a concrete number or an abstract number.

Thus, we can multiply 3 feet by 5, but not 3 feet by 5 feet, nor 5 by 3 feet.

In case a person is reckoning the cost of 5 pounds of tea at 60 cents a pound, he is not multiplying 60 cents by 5 pounds, but is multiplying 60 cents by 5, since he is finding the sum of as many 60 cents as there are pounds.

51. To test the answer, multiply the multiplier by the multiplicand. This should give the same result as multiplying the multiplicand by the multiplier [Theorem I, p. 26].

52. The continued product of a number by itself is called a **Power** of that number.

Thus, 5×5 is called the *second power* or the **square** of 5; $5 \times 5 \times 5$ is called the *third power* or the **cube** of 5; $5 \times 5 \times 5 \times 5$ is called the *fourth power* of 5; and so on.

The squares of the first nine numbers are 1, 4, 9, 16, 25, 36, 49, 64, 81. The cubes are 1, 8, 27, 64, 125, 216, 343, 512, 729.

A small figure placed above and to the right of a number is used to denote a power of that number, and

is called an **Exponent**, or **Index**. For example, 5^4 denotes $5 \times 5 \times 5 \times 5$. Also, $10^2 = 10 \times 10 = 100$, $10^3 = 10 \times 10 \times 10 = 1000$, and $10^6 = 1000000$.

It should be noticed that the second power of 10 contains *two* naughts, the third power *three* naughts, and so on.

When a number is raised to a power, the process is called **Involution**.

Involution, then, is but a name for a special case of *multiplication*.

EXAMPLES VIII.

Oral Exercises.

The pupils should read the answers while looking at the examples.

Multiply each of the following by 10:

1. 23, 2.34, 51.67, 21.08, 700, 314.5.
2. 769, 7.123, .562, 1000.007, .0034.
3. 2.006, .00006, 150.27, 5000000.

Perform the multiplications indicated below:

4. 2.157×100.
5. 1.2308×1000.
6. 1.27×100.
7. 6573×100.
8. $.0067 \times 100$.
9. 2.1345×1000.
10. $.07 \times 1000$.
11. 3.501×1000.
12. $.00039 \times 10000$.
13. 98.764×1000.
14. $.00001 \times 1000$.
15. 16.02×100000.
16. Find the squares of 4, 7, 2, .3, 5.
17. Find 9^2, $.9^2$, 12^2, 1.2^2, $.12^2$.
18. Find 5^2, $.5^2$, $.6^2$, $.7^2$, $.08^2$.
19. Find the cubes of 2, 3, .2.
20. Find $.2^3$, $.02^3$, $.03^3$, $.003^3$.

EXAMPLES IX.

Written Exercises.

Multiply

1. 37 by 3.
2. 65 by 5.
3. 253 by 9.
4. 197 by 8.
5. 384 by 7.
6. 909 by 6.
7. 1083 by 11.
8. 3408 by 12.
9. 597 by 11.
10. 1.6×4.
11. 12.56×17.
12. 142857 by 7.
13. 5.7×8.
14. 31.09×23.
15. 1.25×7.
16. 3.72×9.
17. 92.74×37.
18. 7.314×84.
19. 12345679 by 9.
20. 25 by 20.
21. 27 by 30.
22. 36 by 70.
23. 318 by 50.
24. 527 by 60.
25. 894 by 80.
26. 125 by 700.
27. 389 by 600.
28. 239 by 900.
29. 21670 by 4000.
30. 5790 by 8000.
31. 6175 by 8000.
32. 821400 by 5000.
33. 25 by 25.
34. 27 by 39.
35. 79 by 97.
36. 38 by 56.
37. 79 by 87.
38. 98 by 39.
39. 125 by 47.
40. 384 by 65.
41. 908 by 73.
42. 18 by 12345679.
43. 63 by 12345679.
44. 12.34×2.4.
45. 38.24×3.9.
46. $.1729 \times .24$.
47. $.3462 \times .75$.
48. 3.4165×3.57.
49. $.2675 \times 3.85$.
50. 697 by 123.
51. 587 by 358.
52. 399 by 586.
53. 2809 by 702.
54. 1973 by 904.
55. 3097 by 807.
56. $.0827 \times .2413$.
57. $.0237 \times .5214$.
58. 3156 by 2065.

59. $.13579 \times .0246$.

60. 7802 by 2005.

61. 2.31575×4.0824.

62. 325.1 by 35.79.

63. $.021628 \times .002828$.

64. 13579 by 21695.

65. $.01 \times .01 \times .01$.

66. $.5 \times .05 \times .005$.

67. $6 \times .6 \times .06$.

68. $2.5 \times .25 \times .025$.

69. 3109.72 by 90.706.

70. 823156 by 753698.

71. 826075 by 1509607.

72. 8257.314 by 78167094.

73. 2178 by 5506.

74. 3008 by 2345.

75. 687.4 by .2468.

76. 12837 by 56294.

77. 305.009 by 72809.

78. 123.456 by 65.4321.

79. 5009826 by 7090.068.

80. 21840376 by 9287915.

81. $3.9017 \times .215$.

82. $.00167 \times .0589$.

83. 2.1046×4.0035.

84. $.21089 \times .003904$.

85. $.1 \times .1 \times .1$.

86. $.31 \times .41 \times .51$.

87. 20 by 125.

88. 50 by 350.

89. 800 by 125.

90. 20 by 315.

91. 400 by 125.

92. 600 by 8745.

93. 8000 by 1250.

94. 12000 by 28971.

Find the continued products of

95. 12, 18, and 15.

96. 17, 18, and 19.

97. 3, 4, 5, 6, 7, 8, and 9.

98. 10, 11, 12, 13, and 14.

99. 2, .6, 73, and 5.

100. .7, .3, .006, and 1000.

101. Find $3 \times 7 \times 9 \times 11 \times 13 \times 37$.

Find the squares of

102. 170. **103.** 220. **104.** 360. **105.** 430.

Find

106. 125^2.

107. 5.37^2.

108. 4690^2.

109. 64700^2.

110. 17495^2.

111. 215.729^2.

112. 80^3.

113. 160^3.

114. 800^3.

115. 1600^3.

116. 425^3.

117. 1608^3.

118. 3507^3.

119. 16730^3.

120. $.6^3$.

121. $.76^3$.

122. $.006^3$.

123. There are 2240 pounds in one long ton. How many pounds are there in 517 long tons?

124. There are 168 hours in one week. How many hours are there in 506 weeks?

125. There are 24 sheets in a quire of paper, and 20 quires in a ream. How many sheets are there in 524 reams?

126. There are 86400 seconds in a day. How many are there in 365 days?

127. In an orchard there are 57 rows of gooseberry bushes, and there are 256 bushes in each row. How many bushes are there altogether?

128. A book has 312 pages, on each page there are 32 lines, and in each line there are 42 letters. How many letters are there altogether?

53. It is often convenient to express a number in parts, connected by the sign $+$; thus, we may write $8+4$, or $11+1$, or $7+3+2$, instead of 12.

54. A number expressed in parts may be multiplied just as if expressed as a whole; thus,

$$\begin{array}{r} 6+\ 5 \\ 7 \\ \hline 42+35 \end{array}$$

Here we have 6 units plus 5 units to be multiplied by 7; the answer has the same value as 11×7.

NOTE. A parenthesis may be used to indicate that the several parts compose one number.

Again,

$$\begin{array}{r} 9 \times 4 + 3 \times 8 + 10 \\ 4 \\ \hline 144 \quad + 96 \quad + 40 \end{array}$$

Here we have (9×4) units plus (3×8) units plus 10 units, all to be multiplied by 4; the answer has the same value as 70×4.

$$\begin{array}{r} 3 + 2 \\ 3 + 2 \\ \hline 6 + 4 \\ 9 + 6 \\ \hline 9 + 12 + 4 \end{array}$$

Again, to multiply 5 by 5, or to find 5^2, we may multiply 3 units plus 2 units first by 2 and then by 3 and add the partial products; the answer has the same value as 5×5. It is evident that we may easily square any number between 10 and 100 after separating it into two parts, — its tens and its units. For example, $32^2 =$ the square of 32 after being expressed as the sum of its tens and its units.

$$\begin{array}{r} 30 + 2 \\ 30 + 2 \\ \hline 60 + 4 \\ 900 + 60 \\ \hline 900 + 120 + 4 \end{array}$$

EXAMPLES X.

Written Exercises.

1. Multiply $(9 + 8)$ by 12; $(6 + 11)$ by 8.
2. Multiply $(4 \times 3 + 6)$ by 7; $(9 \times 2 + 8 \times 3)$ by 2.
3. Multiply $(6 + 4 + 2)$ by 3; $(80 + 4 \times 8)$ by 4.
4. Find 13^2; 21^2; 47^2; 94^2; 69^2.

55. The following methods are practical and are of great value in saving time. Pupils should become proficient in performing examples by these methods, and should use them constantly.

I. *Multiplication table for numbers between* 12 *and* 20.

Multiply the units of multiplicand and multiplier, and write the unit figure of the product; then add the tens figure (if any) of the product, the multiplier, and the units of the multiplicand.

Ex. 1.

$$\begin{array}{r} 16 \\ \underline{13} \\ 208 \end{array}$$

MENTAL WORK.

$6 \times 3 = 18$; write 8, then $1 + 13 + 6 = 20$; write 20.

Ex. 2.

$$\begin{array}{r} 13 \\ \underline{1.3} \\ 16.9 \end{array}$$

MENTAL WORK.

$3 \times 3 = 9$; write 9, then $13 + 3 = 16$; write 16.

II. *To multiply by* 11.

Write the unit figure of the multiplicand; then add units and tens, tens and hundreds, etc., separately, writing the right-hand figures of the several sums and carrying the left-hand figures; finally, write the last figure of the multiplicand after adding what was carried.

Ex. 1.

$$\begin{array}{r} 469047 \\ \underline{11} \\ 5159517 \end{array}$$

$$\begin{array}{r} 7 \\ 7 + 4 = 11 \\ 1 \text{ (carried) } + 4 = \ \ 5 \\ 0 + 9 = \ \ 9 \\ 9 + 6 = 15 \\ 1 + 6 + 4 = 11 \\ 1 + 4 = \ \ 5 \end{array}$$

The last figures in the column are the ones to be written in the product.

Ex. 2.

$$\begin{array}{r} 89006.037 \\ \underline{11} \\ 979066.407 \end{array}$$

MENTAL WORK.

7, 10, 4, 6, 6, 0, 9, 17, 9.

Ex. 3.

$$\begin{array}{r} 49.8769 \\ \underline{.11} \\ 5.486459 \end{array}$$

MENTAL WORK.

9, 15, 14, 16, 18, 14, 5.

III. *To square a number of two figures and ending in* 5.

Square the units and write the whole product; then square the tens, add the tens to this square, and write the sum.

Ex. 1.

$$\begin{array}{r} 35 \\ \underline{35} \\ 1225 \end{array}$$

Here $5 \times 5 = 25$
and $3^2 + 3 = 12$.

Ex. 2. $6.5^2 = 42.25$.
For $5^2 = 25$
and $6^2 + 6 = 42$.

EXAMPLES XI.

Multiply

1. 14 by 18.
2. 19 by 13.
3. 1.6 by 1.3.
4. .18 by 1.5.
5. 1.3 by .13.
6. 19 by 19.
7. 1.8 by 18.
8. .16 by .16.
9. 6 by 3 by 14.
10. 2 by 8 by 19.

Multiply the following by 11:

11. 26751.
12. 498.67.
13. 94600.
14. 888.
15. 64.9786.
16. 800960.
17. 493.006.
18. 2.76398.
19. .463.
20. 590001.
21. 9000095.
22. 399678.

Find the following:

23. 55^2; 5.5^2; $.55^2$.
24. 15^2; $.15^2$; 150^2.
25. 45^2; 4.5^2; $.045^2$.
26. 65^2; 7.5^2; 85^2.
27. 25^2; $.25^2$; 250^2.
28. 35^2; 350^2; 3500^2.

This method may be used in finding the squares of 105, 115, and 125.

After a little practice, much of the above work may be done without writing anything but the results.

DIVISION.

56. A short process of finding out *how many equal numbers* may be *together* subtracted from another number is called **Division**.

For example, to divide 12 by 4 is to find out how many fours may be together subtracted from 12, — to find out how many fours there are in 12. The simplest method of finding the required number of fours is to subtract 4 from 12, and then 4 from the remainder, and so on, as many times as possible. It will be found that there is no remainder after subtracting 3 fours. Hence there are 3 fours in 12.

57. The number which is to be thus diminished is called the **Dividend.**

One of the equal numbers to be subtracted is called the **Divisor.**

The result of division is called the **Quotient.**

Ex. 1. *Divide* 24 *by* 8.

24 is the dividend.
8 is the divisor.
3 is the quotient.

Ex. 2. *Divide* 30 *by* 10.

30 is the dividend.
10 is the divisor.
3 is the quotient.

58. Since the dividend contains the divisor as many times as there are units in the quotient, *the dividend is equal to the product of the divisor and the quotient.*

We may say then that *division is the process by which one factor may be found when the product and the other factor (or factors) are given.*

Thus, division is the inverse or undoing of multiplication, just as subtraction is the undoing of addition.

59. Division can be looked upon from two different points of view, the distinction between which is best seen by taking as an example the division of a concrete number.

We have 5 feet $\times$ 7 $=$ 35 feet; and in connection with the undoing of this multiplication, there are the two distinct questions:

(1) How many times is 5 feet contained in 35 feet? The answer to which is 7 *times.*

(2) If 35 feet be divided into 7 equal parts, what will will each part be? Or, what length is contained 7 times in 35 feet?

The answer to which is 5 *feet.*

Thus, in division, either the divisor is an abstract number and the quotient a quantity of the same kind as

the dividend, or else the divisor is a quantity of the same nature as the dividend, and the quotient is an abstract number.

60. Division is indicated by the sign $\div$, which is read, '**divided by**,' or, '**by**.'

Thus, $24 \div 4$ is read 24 divided by 4, and denotes that 24 is to be divided by 4; also, $24 \div 4 \div 3$ denotes that 24 is to be divided by 4 and the result divided by 3, and $24 \div 4 \times 3$ denotes that 24 is to be divided by 4 and the result multiplied by 3.

61. Inexact Division. If we try to divide 14 by 4, we find that after subtracting 3 fours there are 2 units left.

The number left over is called the **Remainder**.

One number is said to be **exactly divisible** by another when it is divisible without remainder.

62. It follows from the definition of division that *the product of the divisor and the quotient plus the remainder is equal to the dividend;* that is,

$$\text{Divisor} \times \text{Quotient} + \text{Remainder} = \text{Dividend}.$$

Hence, if any three of these four numbers be given, the remaining one can be found.

Ex. 1. *The divisor is* 5, *the quotient is* 20, *and the remainder is* 2. *What is the dividend?*

The dividend must exceed the product of the divisor and quotient by 2. Hence,

$$\text{Dividend} = 5 \times 20 + 2 = 102.$$

Ex. 2. *The dividend is* 59, *the quotient* 7, *and the remainder* 3. *What is the divisor?*

The dividend must exceed the product of the quotient and divisor by 3. Hence, the product of the quotient and divisor is $59 - 3 = 56$, and the divisor $= 56 \div 7 = 8$.

63. Division could always be performed by successive subtractions of the divisor, as in Art. 56; but, except in

the case of very small numbers, the process would be extremely tedious, and the necessity for these successive subtractions is obviated by a knowledge of the results of multiplication.

For example, to divide 75 by 9.

Since we know that 8 nines are 72, and that 9 nines are 81, we see that $75 \div 9$ gives 8 for quotient and 3 for remainder.

EXAMPLES XII.

Oral Exercises.

Give the quotient in each of the following cases, and the remainder whenever the division is not exact:

1. $12 \div 4$.	11. $64 \div 8$.	21. $80 \div 9$.
2. $18 \div 9$.	12. $45 \div 9$.	22. $55 \div 9$.
3. $35 \div 7$.	13. $15 \div 4$.	23. $53 \div 7$.
4. $56 \div 8$.	14. $17 \div 5$.	24. $48 \div 5$.
5. $60 \div 10$.	15. $18 \div 7$.	25. $92 \div 9$.
6. $49 \div 7$.	16. $17 \div 3$.	26. $87 \div 8$.
7. $81 \div 9$.	17. $37 \div 9$.	27. $80 \div 7$.
8. $72 \div 8$.	18. $43 \div 5$.	28. $63 \div 5$.
9. $56 \div 7$.	19. $68 \div 7$.	29. $70 \div 6$.
10. $36 \div 6$.	20. $70 \div 8$.	30. $100 \div 9$.

64. Division by 10, 100, etc. To divide any number by 10, it is necessary only to move the decimal point one place to the left. For this divides each of the parts of the number by 10.

For example, 623.4 (see ex., Art. 48) is divided by 10 when its 6 hundreds are made 6 tens, its 2 tens are made 2 units, its 3 units are made 3 tenths, and its 4 tenths are made 4 hundredths; *i.e.*, $623.4 \div 10 = 62.34$.

Also, $62.34 \div 10 = 6.234$; $6.234 \div 10 = .6234$.

Dividing by 10 and by 10 again is the same as dividing by 100, and it will be noticed that in dividing 623.4 by 100 the decimal point is moved *two* places to the left.

Hence, *to divide by* 10, 100, 1000, *etc., move the decimal point as many places to the left as there are naughts in the divisor.*

EXAMPLES.
1. $268706 \div 10 = 26870.6.$
2. $46000 \div 100 = 460.$
3. $26783 \div 1000 = 26.783.$

65. Short Division. — When the divisor is not greater than 12, the process of division can be written in a very compact form. The method will be seen from the following example:

Ex. *Divide* 43251 *by* 8.

The operation is set down in the following form:

$$8\,\underline{)\,43251}$$
5406, remainder 3.

EXPLANATION. First, $43 \div 8$ gives quotient 5 and remainder 3; we put 5 under the 3 of the dividend, as the 5 represents units of the same order as the 3 (namely, *thousands*', in the present case). Then, the remainder 3 is equal to 30 units of the next lower order, and taking into account the next figure of the dividend, namely 2, we have 32 which when divided by 8 gives quotient 4 and 0 remainder; we put down 4 next to 5, and have nothing to 'carry.' Then, $5 \div 8$ gives quotient 0 and remainder 5; we put down 0 next to 4 and 'carry' 5. The 5 carried and 1, the next figure of the dividend, make 51 which when divided by 8 gives quotient 6 and remainder 3. Thus, the complete quotient is 5406 with remainder 3.

EXAMPLES XIII.

Written Exercises.

Divide

1. 92 by 4.
2. 87 by 3.
3. 75 by 5.
4. 234 by 6.
5. 7.85 by 5.
6. 91.8 by 9.

7. 72.15 by 5. 11. 7568 by 11. 15. 823507 by 8.
8. 6.402 by 6. 12. 35.628 by 12. 16. 2104316 by 6.
9. .3564 by 9. 13. 72156 by 9. 17. 123456 by 7.
10. 6822 by 12. 14. 346089 by 7. 18. 987654 by 9.
19. 563753696 by 11. 20. 1374819756 by 12.

Divide without uniting the terms of the dividend

21. $(8 + 14 + 6)$ by 2.
22. $(6 \times 2 + 15 \times 5)$ by 3.
23. $(14 \times 3 + 21 \times 5)$ by 7.
24. $(18 \times 4 + 33 \times 12)$ by 3, and the result by 2.

66. Long Division. — When the divisor is greater than 12 the process of division is written in a long form so that the mind will not become confused.

Ex. 1. *Divide* 1026 *by* 18.

```
18)1026(57
    90
   ---
    126
    126
   ---
```

The full operation may be thus expressed:

```
18)1026(50 + 7
    900
    ---
    126
    126
    ---
```

First, beginning at the left, we use the smallest part of the number that can be divided by 18. Now, neither 1 nor 10 can be divided by 18, but 102 can be. $102 \div 18 = 5$, with a remainder of 12. The quotient 5 is of the same order as the last figure of the dividend used in the first division (just as in short division). The remainder 12 we reduce to units of the next lower order and add the 6 of that order, and we have 126 to be divided by 18. Now $126 \div 18 = 7$.

Ex. 2. *Divide* 102739 *by* 29.

```
29)102739(3542
    87
   ---
    157
    145
    ---
     123
     116
     ---
       79
       58
       --
       21 remainder.
```

The last figure of the dividend used in the first division is 2, and in thousands' place. Therefore the first quotient figure obtained is thousands'.

Ex. 3. *Divide* 44393 *by* 145.

```
145)44393(306
    435
    ---
     893
     870
     ---
      23 remainder.
```

In this example, the first remainder (8 hundreds) reduced to tens and the 9 tens added makes 89 tens, which does not contain 145. Therefore there are no tens in the answer and we write a naught, and proceed by reducing the 89 tens to units, adding 3 units.

Ex. 4.

```
2)16.4
  ----
   8.2
```

The same reasoning applies for a decimal dividend as for an integral dividend. The first figure obtained in the quotient is of the same order as the last figure used in the first division. This fact determines the position of the decimal point.

Ex. 5.

```
9).288
  ----
  .032
```

Ex. 6. *Divide* .019 *by* 125.

```
125).019000(.000152
      125
      ---
       650
       625
       ---
        250
        250
        ---
```

Ex. 7.

```
48)5.220(.108
   4 8
   ---
     420
     384
     ---
      36
```

The remainder is the same in name as the last figure of the dividend. In this case it is .036.

Ex. 8.

```
15)474000(31600
   45
   --
    24
    15
    --
     90
     90
     --
```

In this example, it is unnecessary to extend the written work beyond dividing 90 hundreds by 15. Since, however, every order of the dividend must have a corresponding figure in the quotient, we write naughts in tens' and units' places.

67. *To divide, when the divisor is partly or wholly a decimal.*

Here we make use of the following principle:

Multiplying both dividend and divisor by the same number does not change the quotient.

Thus, $24 \div 4 = 6$, and if both 24 and 4 be multiplied by 2, we shall have $48 \div 8 = 6$; also, $3.6 \div .6 = 6$, and if both 3.6 and .6 be multiplied by 10, we shall have $36 \div 6 = 6$.

Hence, to divide any number by a decimal, we first **multiply both dividend and divisor by that power of 10 which will make the divisor a whole number, and then proceed as in the case of division by a whole number.** We perform these multiplications by moving the decimal points.

Ex. 1. *Divide* 11.68 *by* 1.6

Move the decimal points one place to the right. Then

```
1.6,)11.6,8(7.3
     11 2
     ----
        4 8
        4 8
        ---
```

Do not forget that **the first significant figure of the quotient is of the same order as the last of those figures of the dividend which are used in the first division. This will indicate the position of the decimal point in the quotient.**

Ex. 2. *Divide* .21 *by* .0125

```
.0125,).2100,0(16.8
        125
        ---
         850
         750
         ---
         100 0
         100 0
         -----
```

Ex. 3. *Divide* .0697585 *by* 1.33.

```
1.33,).06,97585(.05245
          6 65
          ----
           325
           266
           ---
            598
            532
            ---
             665
             665
             ---
```

NOTE. *Always* let the old decimal point remain, and indicate the new one by a mark similar to those in Ex. 1.

68. NOTE. It should be noticed that, although the *quotient* is unchanged by multiplying both dividend and divisor by the same number, the *remainder*, if any, is not unchanged, but is equal to the original remainder multiplied by the number by which the original divisor and dividend were multiplied.

For example, $26 \div 6 = 4$, with a remainder of 2; and 8 times 26 divided by 8 times 6 equals 4, with a remainder of 8 times 2. Therefore, we must divide the remainder by the multiplier, if we wish the remainder obtained by using the original numbers, as the remainder is the part of dividend not used.

Ex. 1. *Divide* 17.8 *by* 1.4.

```
1.4)17.8(12
    14
     3 8
     2 8
Remainder 1 0 units.
```

The remainder would have been 1 unit if we had not multiplied by 10.

Ex. 2. *How many pieces each* 1.02 *inches long can be cut from a rod whose length is* 18 *inches?*

We can find the *quotient* by dividing 1800 by 102. Thus

```
102)1800(17
    102
     780
     714
      66
```

Hence there are 17 pieces; and since the original divisor and dividend were multiplied by 100, the remainder left over is

$$(66 \div 100) \text{ inches} = .66 \text{ inches}.$$

69. Division by Factors. — We have seen that to multiply by two or more numbers in succession gives the same result as to multiply at once by their product. It therefore follows, conversely, that to divide by two or more numbers in succession gives the same result as to divide at once by the product of the numbers.

Ex. 1. *Divide* 11445 *by* 35.

Since $35 = 7 \times 5$, we may divide by 7 and 5 in succession.

$$\begin{array}{r} 7\underline{)11445} \\ 5\underline{)1635} \\ 327 \end{array}$$

Ex. 2. *To divide* 315637 *by* 20.

$$\begin{array}{l} 2,0\underline{)31563,7} \\ \quad\ 15781, \text{ remainder } 17. \end{array}$$

Dividing both dividend and divisor by 10, as indicated, we have 31563 to be divided by 2. The quotient is 15731 and the remainder 1, which must be multiplied by 10 and the figure cut off by the decimal point annexed, making 17 as the true remainder.

70. When one number is divided by several others in succession, the method of finding the remainder will be seen from the following example:

Ex. 1. *Divide* 11467 *by* 35.

$$\begin{array}{l} 7\underline{)11467} \\ \quad 5\underline{)1638} \textit{ sevens} \text{ and 1 unit over.} \\ \qquad\quad 327 \textit{ thirty-fives} \text{ and 3 } \textit{sevens} \text{ over.} \end{array}$$

The whole remainder is therefore 3 sevens and 1 unit, that is, 22.

From the above it will be seen that the whole remainder is found by *multiplying the remainder after the second division by the first divisor and then adding the remainder after the first division.*

Ex. 2. *Divide* 251633 *by* $3 \times 5 \times 7$.

$$\begin{array}{l} 3\underline{)251633} \\ \ 5\underline{)83877} \text{ groups of 3 each and 2 units over.} \\ \ 7\underline{)16775} \text{ groups of } 3 \times 5 \text{ each and 2 groups of 3 each over.} \\ \qquad 2396 \text{ groups of } 3 \times 5 \times 7 \text{ each and 3 groups of } 3 \times 5 \text{ each over.} \end{array}$$

The whole remainder is therefore 3 groups of 3×5 each $+ 2$ groups of 3 each $+ 2 = 3 \times 3 \times 5 + 2 \times 3 + 2 = 45 + 6 + 2 = 53$.

Thus, if there are more than two successive divisions the whole remainder is found by multiplying each remainder by all the divisors preceding that from which the remainder arises, and then adding these results to the first remainder.

71. The work of finding some *products* may be shortened by making use of multiplication and division at the same time.

Ex. 1. *Multiply* 6174 *by* 25.

Since $25 = 100 \div 4$, we shall multiply by 25 if we first multiply by 100 and then divide by 4. For by multiplying by 100 we get 4 times too much, which is put right when we divide by 4. To multiply by 25 we may therefore affix two naughts and divide by 4; thus,

$$\begin{array}{r} 4\underline{)617400} \\ 154350 \end{array}$$

Ex. 2. *Multiply* 6174 *by* 125.

Since $125 = 1000 \div 8$, we multiply by 1000 and then divide by 8, that is, we affix three naughts and divide by 8; thus,

$$\begin{array}{r} 8\underline{)6174000} \\ 771750 \end{array}$$

The methods adopted in the following examples are also worth notice.

Ex. 3. *Multiply* 7964 *by* 9998.

Since $9998 = 10000 - 2$, we can multiply by 10000 and by 2, and take the difference of these products.

$$\begin{array}{r} 7964 \\ \underline{9998} \\ 79640000 \\ \underline{15928} \\ 79624072 \end{array}$$

Ex. 4. *Multiply* 7.964 *by* 9998.

$$\begin{array}{r} 7.964 \\ \underline{9998} \\ 79640. \\ \underline{15.928} \\ 79624.072 \end{array}$$

72. To test the answer in division, multiply the quotient by the divisor (not divisor by quotient), and to the product add the remainder (if any); the result should equal the dividend. [Art. 58.]

73. Some saving of time in division will be effected by performing the multiplication of the divisor and the subtraction from the dividend simultaneously; this method should, however, be attempted only by those who show some aptitude for numerical calculations, for the slight gain in speed by no means makes up for the increased liability to error.

The method will be understood from the following example:

Divide 102739 *by* 29.

```
29)102739(3542
     157
      123
        79
        21 rem.
```

EXPLANATION. Instead of multiplying 29 by 3 and subtracting the whole product from 102, we subtract the several figures of the product as we go along. Thus, 3 times 9 are 27, and 7 from 12 leaves 5; we write 5, and carry 3 (2 from the 27, and 1 from the 12). Then, 3 times 2 are 6, and 3 (carried) are 9, and 9 from 10 leaves 1. The remainder is 15, which with the 7 of the dividend makes 157 for the next partial dividend. And so on to the end.

EXAMPLES XIV.

Written Exercises.

a.

Divide

1. 182 by 13.	4. 399 by 19.	7. 702 by 26.
2. 204 by 17.	5. 575 by 23.	8. 1054 by 34.
3. 221 by 17.	6. 899 by 29.	9. 4185 by 31.

10. 1591 by 37.	14. 430686 by 71.
11. 6016 by 94.	15. 415242 by 59.
12. 710007 by 87.	16. 426713 by 47.
13. 435435 by 65.	17. 562171 by 53.

18. 850902 by 78.
19. 1173021 by 97.
20. 1034550 by 95.
21. 2706420 by 86.
22. 11336 by 109.
23. 22563 by 207.
24. 160335 by 315.
25. 39483 by 123.
26. 21112 by 104.
27. 185745 by 305.
28. 801738 by 567.
29. 8035370 by 2674.
30. 9570744 by 1593.
31. 407514744 by 6724.
32. 31587678 by 5067.
33. 266 + 126 + 210 by 14.
34. 6164 + 5226 by 67.

35. The trees in an orchard are arranged in 153 rows, with the same number of trees in each row, and there are 16371 trees altogether. How many trees are there in each row?

36. There are 86400 seconds in a day; in how many days are there 13564800 seconds?

b.

In division of decimals, the quotient should be continued until there is no remainder, unless otherwise directed. This can be accomplished by annexing naughts to the dividend, as in Ex. 6, Art. 66. In general practice three or four decimal places in the quotient are considered sufficient.

Divide

1. 16.4 by 2.
2. 32.7 by 3.
3. 17.2 by 4.
4. .156 by 6.
5. .288 by 9.
6. .135 by 9.
7. 125.6 by 20.
8. 31.83 by 30.
9. 11.7215 by 50.
10. 215.4 by 80.
11. .0321 by 60.
12. .174 by 120.
13. 5.22 by 48.
14. .171 by 72.
15. .012 by 1600.
16. .027 by 45.
17. 2.355 by 75.
18. 2.715 by 48.

19. 52.7 by 17.
20. 43.7 by 23.
21. 166.6 by 119.
22. 3751.5 by 123.
23. 3.7515 by 1230.
24. 375.15 by 125.
25. 37.515 by 1250.
26. 67.77 by 135.
27. .006777 by 1350.
28. 1.036656 by 207.
29. .1036656 by 5008.
30. .001036656 by 2070.
31. .651714 by 3156.

Find, to 4 places of decimals,

32. $12.15 \div 148$.
33. $2.374 \div 156$.
34. $41.75 \div 89$.
35. $135 \div 17$.
36. $17 \div 135$.
37. $121 \div 170$.

38. Simplify $.026 \times .0493 \div 221$.

c.

Divide

1. 6.2 by .01.
2. .347 by .001.
3. 12.3 by .0001.
4. 3.5 by .5.
5. .75 by .05.
6. 1.25 by .005.
7. 62.5 by 2.5.
8. .625 by .025.
9. 625 by .0025.
10. 1.1 by .125.
11. .019 by 1.25.
12. 170 by .00125.
13. 1.5 by 2.4.
14. 5.76 by 4.8.
15. 8.1 by .36.
16. 159.1 by 3.7.
17. 6.016 by .94.
18. 70.992 by 8.7.
19 .435435 by .0065.
20. 430.686 by .0071.
21. 415.242 by .0059.
22. .185745 by 3.05.
23. 4.07514744 by .006724.
24. .9570744 by 159.3.

25. Find, to 4 places of decimals, $43.21 \div 123.4$, and $.0167 \div 3.17$.

26. Simplify $360 \div 7.2 \div .16$.

27. Simplify $.0441 \div .21 \div .56$.

28. Simplify $1.953 \div 8.68 \times .035$.

29. How many lengths each 2.56 inches are there in a rod 120 inches long; and how much is left over?

30. How many packets of tea, each containing 1.85 ounces, can be made up out of a chest containing 2400 ounces; and how much is left over?

d.

Divide, using factors not greater than 12,

1. 396 by 18.
2. 816 by 24.
3. 625 by 25.
4. 3753 by 27
5. 8820 by 36.
6. 15750 by 42.
7. 1958528 by 64.
8. 59081805 by 81.
9. 13339728 by 108.
10. 10654069140 by 132.
11. 316794 by 45.
12. 7196243 by 35.
13. 2106935 by 36.
14. 9172143 by 72.
15. 22222222 by 99.
16. 123456789 by 132.
17. 32163 by 20, 30, and 40.
18. 21574 by 20, 40, and 60.
19. 123456 by 20, 30, and 40.
20. 158937 by 20, 50, and 70.
21. 2167 by 30, and 50.
22. 16819 by 30, and 80.
23. 17943 by 40, and 60.
24. 21985 by 50, and 90.
25. 217943 by 500.
26. 712415 by 700.
27. 217643 by 216.
28. 1234567 by 242.

e.

Multiply, using the short process,

1. 74562 by 25.
2. 4.162 by 25.
3. 12678 by 125.
4. 387.4 by 125.
5. 79.624 by 99.
6. 1897 by 999.

7. 29075 by 998.
8. .79184 by 9999.
9. 6729 by 12.5.
10. .6003 by 12.5.
11. 786 by 250.
12. 34.65 by .0125.

EXAMPLES XV.

Miscellaneous Examples, Chapters I and II.

1. Express in words 3015602, and in figures eleven million five hundred thousand two hundred fourteen.

2. Find the sum of 30157, 12.468, 31947, and 3.6539.

3. By how many is 13018 greater than 12997?

4. Multiply 8000 by 1250, and 3200 by 12345.

5. How many times can 317 be subtracted from 1389, and what is the remainder?

---

6. Express MDCCCLXXIX in the Arabic notation, and 1449 by means of Roman numerals.

7. Find $1325 + 3016 + 79 + 90167$.

8. Find $316 - 179 + 257 - 89 - 185 + 398 - 485$.

9. Multiply 1234 by 4321 and 9009 by 31562.

10. How many nineteens are there in five thousand, and how many are over?

11. By how much does the sum of 3.72 and 10.015 fall short of the sum of 7.216 and 6.52?

---

12. Express in words 1632057 and 3004167201500.

13. Subtract the sum of 3158, 2016, and 5143 from 11111.

14. Multiply the difference between seventy-six million seventy-six and four hundred forty thousand four hundred forty, by eleven hundred fourteen.

15. A farmer has 197 sheep and three times as many lambs. How many sheep and lambs has he altogether?

16. Find by short divisions how many thirty-fives there are in 31578, and how many are over.

17. Add 31.057, 156.0083, 2.61759, and .008347.

18. Subtract the difference between 3.14 and 1.0625 from the sum of 1.00172 and 2.127.

---

19. By how many is one million eight thousand nine hundred seventy-four less than two million eleven hundred twelve?

20. Find $3142 - 1250 - 989 + 6217 - 3587 - 1924$.

21. A farmer had 2000 bags of wheat. He sold 527 bags to one man and 255 bags to each of three others. How many bags were left unsold?

22. How many letters are there in a book of 375 pages, each page of which contains 32 lines, and each line 45 letters?

23. Multiply 31.025 by .032, and .0625 by .00125.

24. By what number must 59755 be divided in order that the quotient may be 19?

25. Divide 7.0175 by 17.5, and 7.5 by .00625.

---

26. In one school there are one hundred seventy-six boys and one hundred and twelve girls; and in another school there are half as many boys and twice as many girls. How many scholars altogether are there in the two schools?

27. The sum of two numbers is 317205 and one of them is 185964; what is the other?

28. A farmer sold 75 cattle at 24 dollars a head and bought with the money sheep at 2 dollars each. How many sheep did he buy?

29. Divide .04312 by .0044, and 9.0225 by .225.

30. Divide 358 by 15 by short divisions.

31. What is the least number which must be added to 57914 in order that the sum may be exactly divisible by 315?

32. Divide the product of 37.5 and .1248 by .005625.

---

33. Express MDCCCXCIV in the Arabic notation, and 2875 by means of Roman numerals.

34. In a school of four hundred and ninety children there are two hundred and seventy-six girls. How many more girls than boys are there?

35. In a train there are 37 cars each having seats for 36 people, and there are 375 passengers in the train; how many seats are empty?

36. Simplify $1.702 \times 2.9015 \div .0005803$.

37. Divide the product of .0374 and .0075 by the difference between .675 and .6375.

38. Show that the sum of the squares of three thousand nine, and four thousand twelve, is equal to the square of five thousand fifteen.

39. What is the least number which must be subtracted from 2146537 in order that the remainder may be exactly divisible by 4275?

---

40. Subtract nine hundred five million eight thousand nine hundred sixty-five from eleven hundred million two thousand three hundred, and express the result in words.

41. At an election, the successful candidate, who obtained 12597 votes, had a majority of 1479 over the unsuccessful candidate. How many votes were given altogether?

42. Find $2197 - 1982 + 374 + 10085 - 8216 + 11597 - 7986$.

43. Find the squares of 2.15 and .0324.

44. Multiply 16777216 by 131072, also divide 16777216 by 131072, and express the results in words.

45. Find the least number of repetitions of 3745 whose sum is greater than a million.

46. Divide .378 by 262.5, and 37.8 by .02625.

---

47. Express the numbers 29, 47, 158, 679, 1464, and 10385 by means of Roman numerals.

48. How many figures are there in all the numbers from 1 to 100? How many in the numbers from 1 to 1000?

49. A certain number when divided by 3008 gives a quotient 3875 and a remainder 2794. What is the number?

50. Divide 999999 by the continued product of 3, 7, 11, and 13.

51. The sum of two numbers is 315642, and one of the numbers is twice the other: find them.

52. Divide 2722.05 by .345, and .0272205 by 3.45.

53. Divide $(144.4 + 152 \times 4.6)$ by 19; prove your answer by dividing after uniting the terms of the dividend.

54. Divide, by factors, $(6.3 \times 6 + 4.9 \times 18)$ by 21.

See Art. 44 for definition of factors.

CHAPTER III.

FACTORS AND MULTIPLES—SQUARE ROOT—HIGHEST COMMON FACTOR—LEAST COMMON MULTIPLE.

FACTORS.

74. AN exact divisor of a number is called a **Factor** of that number; thus,

2, 3, 4, 6, and 12 are factors of 24. [Art. 44.]

A factor is also called a **Measure.**

A number that is exactly divisible by another number is called a **Multiple** of that number; thus,

12, 30, 54, 72, and 90 are *multiples* of 6.

It will be seen at once that a number has a limited number of factors, but an unlimited number of multiples.

75. A number which is not divisible by any number except itself and 1 is called a **Prime Number,** or a **Prime.**

Thus, 2, 3, 5, 7, etc., are *primes*.

Every number which has other factors beside itself and unity is called a **Composite** number.

Thus, 4, 6, 8, 9, etc., are *composite* numbers.

Two numbers, both of which cannot be divided by the same number (except unity), are said to be **prime to one another.**

Thus, 4 and 9 are prime to one another; both, however, are composite numbers.

76. Numbers divisible by 2 are called **Even** numbers. Numbers not divisible by 2 are called **Odd** numbers.

2, 14, 30, and 74 are even numbers.
3, 7, 27, and 51 are odd numbers.

The following simple conditions of divisibility will be found to be useful:

(i) A number whose last digit expresses an *even* number is divisible by 2.

248 and 100694 are divisible by 2.

(ii) A number whose last digit is 5 or 0 is divisible by 5.

25, 55, and 600 are divisible by 5.

(iii) A number whose last two digits express a number divisible by 4 or by 25 is divisible by 4 or by 25, respectively.

67215736 is divisible by 4.
23798675 is divisible by 25.

(iv) A number the sum of whose digits is divisible by 3 or by 9 is divisible by 3 or by 9, respectively.

The sum of the digits of the number 56174154, namely,

$$5 + 6 + 1 + 7 + 4 + 1 + 5 + 4, \text{ is } 33;$$

and 33 is divisible by 3, but is not divisible by 9. Thus, the number 56174154 is divisible by 3, but not by 9.

(v) A number is divisible by 11 when the difference between the sum of the first, third, fifth, etc., digits and the sum of the second, fourth, sixth, etc., digits is zero or a multiple of 11, and not otherwise.

Thus, 3572129 is seen to be divisible by 11, since $9 + 1 + 7 + 3$ differs from $2 + 2 + 5$ by 11.

EXAMPLES XVI.

Oral Exercises.

Which of the numbers, 2, 4, 8, 3, 9, 5, 25, 125, 11, can be seen by inspection to be factors of

1. 964.	4. 7326.	7. 94680.	10. 49125.
2. 225.	5. 6975.	8. 29304.	11. 307890.
3. 1925.	6. 4125.	9. 76164.	12. 264792.

77. The following are important general theorems:

I. *Every divisor or factor of each of several numbers is a divisor of their sum.*

If, for example, each of several numbers is divisible by 12, then each can be arranged in groups of *twelve*, and therefore their sum consists of a certain number of *twelves*. Similarly for any other divisor.

II. *Every divisor of a number is a divisor of any multiple of that number.*

If, for example, any number is divisible by 12, it can be arranged in groups of *twelves*, and so also can any number of repetitions of the number.

III. *Every divisor of two numbers is a divisor of the sum, or of the difference, of any multiples of the numbers.*

If, for example, two numbers are both divisible by 12, they can both be arranged in groups of *twelves*, and so also can any multiples of either. These multiples can then be added, or one can be taken from the other, *without taking to pieces any of the groups.*

To make the above theorems quite clear to a beginner, it would be well to have actual counters to deal with, which could be tied up by *twelves* in bags or bundles. The pupil would then see that the different additions and subtractions could be performed *without undoing any*

of the bags or bundles, and therefore the final result must be a certain number of *twelves.*

78. The Sieve of Eratosthenes. — The different prime numbers can be found in order by the following method, called the Sieve of Eratosthenes.

Write in their natural order the numbers from 1 to any extent that may be required; thus,

1,	2,	3,	$\dot{4}$,	5,	$\ddot{6}$,	7,	$\dot{8}$,	$\dot{9}$,	$\ddot{10}$,
11,	$\ddot{12}$,	13,	$\ddot{14}$,	$\ddot{15}$,	$\dot{16}$,	17,	$\ddot{18}$,	19,	$\ddot{20}$,
$\ddot{21}$,	$\dot{22}$,	23,	$\ddot{24}$,	$\dot{25}$,	$\dot{26}$,	$\dot{27}$,	$\ddot{28}$,	29,	$\dddot{30}$,
31,	$\dot{32}$,	$\dot{33}$,	$\dot{34}$,	$\ddot{35}$,	$\ddot{36}$,	37,	$\dot{38}$,	$\dot{39}$,	$\ddot{40}$, etc.

Now take the first prime number, 2, and over every second number from 2 place a dot: we thus mark all the multiples of 2. Then, leaving 3 unmarked, place a dot over every third number from 3: we thus mark all multiples of 3. The number next to 3 left unmarked is 5; and, leaving 5 unmarked, place a dot over every fifth number from 5: we thus mark all multiples of 5. And so for multiples of 7, etc.

By proceeding in this way all multiples of the prime numbers, 2, 3, 5, 7, etc., are struck out; also multiples of all composite numbers are necessarily struck out at the same time: for example, all multiples of 6 are struck out as being multiples of either of its prime factors 2 or 3. Hence all the numbers which are left unmarked are primes, for no one of them is divisible by any number (except unity) which is smaller than itself.

We can thus find in order as many prime numbers as we please.

The primes less than 100 will be found to be

1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41,
43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.

79. To find whether a given number is or is not a prime, we have only to see whether it is divisible by any one of the *prime* numbers, 2, 3, 5, 7, etc.

Ex. 1. *Is* 233 *a prime number?*

By trial it will be found that 233 is *not* divisible by 2, nor by 3, nor by 5, nor by 7, nor by 11, nor by 13, nor by 17. Now it is not necessary to try any other primes, for $233 \div 17$ gives a quotient less than 17; if, therefore, 233 were divisible by a prime greater than 17, the quotient would be less than 17, and 233 would be divisible by this quotient, that is by a number less than 17, which we know is not the case. Hence 233 is a prime number.

80. Resolution into Prime Factors.

The following examples will suffice to show how to express any number whatever as the product of factors each of which is a prime.

The method is applicable to all numbers however large, provided we find as many prime numbers as may be necessary by means of the 'sieve'; the method would, however, be extremely tedious in the case of a very large number.

Ex. 1. *Express* 28028 *as the product of prime factors.*

$$\begin{aligned} 28028 &= 2 \times 14014 \\ &= 2 \times 2 \times 7007 \\ &= 2 \times 2 \times 7 \times 1001 \\ &= 2 \times 2 \times 7 \times 7 \times 143 \\ &= 2 \times 2 \times 7 \times 7 \times 11 \times 13. \end{aligned}$$

These continuous divisions may be thus expressed:

$$\begin{array}{r|r} 2 & 28028 \\ \hline 2 & 14014 \\ \hline 7 & 7007 \\ \hline 7 & 1001 \\ \hline 11 & 143 \\ \hline & 13 \end{array}$$

Ex. 2. *Find the prime factors of* 3978.

$$\begin{array}{r|r} 2 & 3978 \\ \hline 3 & 1989 \\ \hline 3 & 663 \\ \hline 13 & 221 \\ \hline & 17 \end{array}$$

The answer is 2, 3^2, 13, and 17.

Ex. 3. *Obtain two factors of* $14 + 22$.

$$14 + 22 = 2 \text{ multiplied by } (7 + 11).$$

EXAMPLES XVII.

Express the following numbers as products of prime factors:

Oral Exercises.

1. 6, 9, 10, 15, 24, 30, 36, 39, 45, 48.
2. .6, .9, 1.5, 2.4, 3.6, 3.9, 4.5, 4.8.
3. .09, .15, .24, .36, .39, .45, .48.
4. 49, 50, 54, 60, 5.4, 75, 81.
5. 3.2, 100, 120, 130.

Written Exercises.

6. 184, 196, 275, 273, 391, 525.
7. 350, 459, 715, 728, 792, 999.
8. 1092, 3885.
9. 51051, 74613, 462462.
10. Obtain two factors of $(6 + 15)$.
11. Obtain three factors of $(30 + 70)$.
12. Obtain two factors of $(2 \times 6 + 4 \times 5 + 2 \times 17)$.

SQUARE ROOT.

81. Obtain the two equal factors of 4; of 9; of 25; of 0.4.

Obtain the three equal factors of 8; of 27; of .008.

Obtain the four equal factors of 16; of 81.

One of the equal factors of a number is called a **Root** of the number; thus, 3 is a root of 9; 5 is a root of 25; 3 is a root of 27; .2 is a root of .04; .2 is a root of .008.

If a number is the product of *two* equal factors, its root is called a *second* root, or **Square Root.**

If a number is the product of *three* equal factors, its root is called a *third* root, or **Cube Root.**

Likewise we have *fourth* and *fifth* roots, etc.

82. It was shown in Art. 52 that a *square* is obtained when the multiplicand equals the multiplier.

Here it is seen that a *square root* is obtained when the quotient equals the divisor.

83. The squares of the first 12 whole numbers should be known: they are

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144.

It will be seen at once that the square root of an integer is by no means always an integer; in fact the only numbers between 1 and 100 which have an integral square root are 4, 9, 16, 25, 36, 49, 64, and 81.

It will be seen later on that the square root of an integer which is not the square of a whole number can be found approximately only.

An integer (or a decimal) which is the square of another integer (or decimal) is called a **Perfect Square.**

Thus, 16 and .09 are perfect squares; namely, the squares of 4 and .3, respectively.

84. The sign $\sqrt{\ }$ is used to indicate a root, and is called the **Radical Sign.**

If any other root than the second is to be indicated, a small figure called an **Index** is placed just above the radical sign; thus,

$\sqrt{9}$ indicates the square root of 9;

$\sqrt[3]{8}$ indicates the cube root of 8;

$\sqrt[5]{243}$ indicates the fifth root of 243.

85. In simple cases, the *square root* of a given number can be found by separating it into factors which are squares, and making use of the principle that *the product of the squares of two or more quantities is equal to the square of the product of those quantities.*

For example, to find $\sqrt{324}$.

$$324 = 4 \times 81 \qquad = 2^2 \times 9^2 = (2 \times 9)^2;$$

hence, $\sqrt{324} = \sqrt{(2 \times 9)^2} = 2 \times 9 \quad = 18.$

Also, $\sqrt{1.44} = \sqrt{(2^2 \times .6^2)} = \sqrt{(2 \times .6)^2} = 2 \times .6 = 1.2.$

EXAMPLES XVIII.

Written Exercises.

Find the square roots of the following numbers:

1. 196; 1.96.	**6.** 576; 5.76.	**11.** 2601.
2. 225; 2.25.	**7.** 676; 6.76.	**12.** 3969; .003969.
3. 324.	**8.** 1089; .1089.	**13.** 4225; 42.25.
4. 400; 4.84.	**9.** 1225; 12.25.	**14.** 7056.
5. 441; 4.41.	**10.** 2025.	**15.** 11025.

In each of the following numbers, what is the least multiplier that will produce a perfect square?

16. 12.	**18.** 24.	**20.** 126.	**22.** 1176.
17. 20.	**19.** 52.	**21.** 140.	**23.** 1344.

24. State a number which has a second and a fourth root; a second, third, and sixth root.

86. The above method cannot be easily used in all cases, but the method which can be used will be understood from the following explanation. [Arts. 86, 87, 88.]

Let it be required to find 63^2. This may be done in the usual way, and the square is found to be 3969.

Now 63^2 may be written $(60 + 3)^2$, which equals the square of 60 + twice the product of 60 by 3 + the square of 3.

$$\begin{array}{r} 60 + 3 \\ 60 + 3 \\ \hline 60 \times 3 + 3^2 \\ 60^2 + 60 \times 3 \qquad \\ \hline 60^2 + 2(60 \times 3) + 3^2 \end{array}$$

The square of the sum of any other pair of numbers can be expressed in a similar form.

Hence, **the square of the sum of any two numbers is equal to the sum of their squares plus twice their product.**

87. Since

$$.01^2 = .0001, \qquad 10^2 = 100,$$
$$.1^2 = .01, \qquad 100^2 = 10000,$$
$$1^2. = 1, \qquad 1000^2 = 1000000,$$

and so on, it follows that if a number has one digit, its square has either one or two digits; if a number has two digits, its square has either three or four digits; if a number has three digits, its square has either five or six digits; and so on.

Hence, if we mark off the digits of a given number, beginning at the units' digit, into periods of two, the last of the periods on the left containing either one or two digits; then *the number of these periods will be equal to the number of digits in the square root of the given number.*

For example, by pointing off the numbers, 961, 54.76, 36.8449, 1522756, thus, 9′61, 54′.76, 36′.84′49, 1′52′27′56, we see that the square roots of these numbers contain 2, 2, 3, and 4 figures, respectively.

88. To find the Square Root of Any Number.

The method will be seen from the following examples:

Ex. 1. *To find the square root of* 3969.

$$\begin{array}{r|l} & 39'69'(60+3 \\ & \underline{36\,00} \\ 2\times 60+3=123) & 3\,69 \\ & \underline{3\,69} \end{array}$$

By pointing off the digits into periods of two, we see that there are *two* digits in the required root; and, since $60^2 = 3600$ and $70^2 = 4900$, we see that the root lies between 60 and 70. The tens' digit must therefore be 6, and we have now to find the units' digit.

If we subtract 60^2 from the given number, the remainder is 369; and, by Art. 86, this remainder is equal to (2×60) times units' digit + (units' digit)^2, or units' digit times $(2 \times 60 + \text{units' digit})$;

i.e., 369 is the product of the unknown digit by $(2 \times 60 +$ the unknown digit).

Hence, if we use 2×60 as a **trial divisor**, we obtain a quotient, namely 3, which is either equal to or *greater* than the required digit. Put this quotient for the unknown digit, and we have $(2 \times 60 + 3)$, or 123, as a true, or complete, divisor. Now dividing 369 by 123, we find that 3 *is* the correct digit for units' place.

```
    39'69'(63
    36
123)369
    369
```

The process is shortened, as in ordinary division, by the omission of zeros; the periods, of two figures each, are brought down one at a time, one figure of the root corresponding to each period.

Ex. 2. *Find the square root of* 114244.

```
          11'42'44'(300 + 30 + 8             11'42'44(338
           9 00 00                            9
600 + 30)2 42 44                          63)2 42
         1 89 00                             1 89
  660 + 8) 53 44                         668) 53 44
           53 44                              53 44
```

There are here three periods and therefore three digits in the root, the first of which is 3, since 114244 is between 300^2 and 400^2. Using 300×2 as a trial divisor in order to find the second figure in the root, we obtain the quotient 40; this, however, is too great, for $(600 + 40)$, the complete divisor, is not contained 40 times in the dividend; we therefore try 30, which proves to be correct.

The process is usually indicated in the shortened form, any *trial* divisor being the product of the quotient already found by 2 and 10 continuously, while the corresponding *complete* divisor is the trial divisor with its naught displaced by the quotient figure obtained in using the trial divisor: thus, in Ex. 2, the first trial divisor is $3 \times 2 \times 10 = 60$, while the complete divisor is 63; also the second trial divisor is $33 \times 2 \times 10 = 660$, while the complete divisor is 668.

Ex. 3. *Find the square root of* 50126400.

```
     50'12'64'00'(7080
     49
1408)1 12 64
     1 12 64
           00
```

Here there are four periods and therefore four figures in the root. A figure of the root corresponds to each period brought down in the shortened process; and in the present case two figures of the root are naughts.

Ex. 4. *Find* $\sqrt{14.44}$.

In the case of a decimal, the pointing must be begun at the decimal point, and carried to the left for the integral part, and to the right for the decimal part.

```
     14.'44'(3.8
      9
68)5 44
   5 44
```

Ex. 5. *Find* $\sqrt{315}$.

```
     3'15.'00'00'(17.74 +
     1
27)2 15
   1 89
347) 26 00
     24 29
3544) 1 71 00
      1 41 76
         29 24
```

Having used both periods of the given number, there is a remainder of 26. We place a decimal point after the units' figure of both dividend and quotient, and then continue the periods by using naughts. The process would never terminate, hence 315 is not a perfect square. We obtain, however, an approximate answer by stopping after the second or third decimal place.

89. Since the square of a number cannot end with a naught unless the number itself ends with a naught, it follows that, if the process of finding a square root does not terminate when the last significant figure is brought down, the process will never terminate.

Expressions such as $\sqrt{3}$, $\sqrt{2.5}$, which cannot be found *exactly* are called **Surds**.

Although no definite number can be found whose square is *exactly* equal to 3, the process of Art. 88, Ex. 5, *if continued far enough*, will enable us to find a decimal whose square differs from 3 by as small a quantity as we please.

EXAMPLES XIX.

Written Exercises.

Find the square roots of

1. 729.	4. .1849.	7. 16.81.
2. 3481.	5. 2209.	8. 56169.
3. 11.56.	6. 6084.	9. 4157521.

10. 49126081.	**13.** 9345249.	**16.** 13.69.
11. 26625600.	**14.** 934.5249.	**17.** 136.9.
12. 182.493081.	**15.** 1369.	**18.** 1.369.

19. .00022201. **20.** 2.2201.

Find, to three decimal places,

21. $\sqrt{5}$.	**24.** $\sqrt{125.4}$.	**27.** $\sqrt{.081}$.
22. $\sqrt{19}$.	**25.** $\sqrt{31.046}$.	**28.** $\sqrt{.01735}$.
23. $\sqrt{21.5}$.	**26.** $\sqrt{.4}$.	**29.** $\sqrt{.0002}$.

HIGHEST COMMON FACTOR.

90. A number which *exactly* divides two or more numbers is called their **Common Factor.**

For example, 2, 3, and 6 are common factors of 18 and 24.

The largest number which *exactly* divides two or more numbers is called their **Highest Common Factor** (H.C.F.); called, also, the **Greatest Common Measure** (G.C.M.), and the **Greatest Common Divisor** (G.C.D.).

Thus, 6 is the H.C.F. of 18 and 24,
or the G.C.M. of 18 and 24,
or the G.C.D. of 18 and 24.

91. After numbers have been resolved into their *prime* factors, their *H.C.F.* can be found by inspection.

Consider, for example, the numbers 30 and 42.

$$\begin{aligned} 30 &= 2 \times 3 \times 5 \\ 42 &= 2 \times 3 \times 7 \\ \hline \text{H.C.F.} &= 2 \times 3 \\ &= 6. \end{aligned}$$

Here we see that 2 and 3 are the only primes that are divisors of both 30 and 42. Therefore the H.C.F. $= 2 \times 3 = 6$.

Again,

$$\begin{aligned} 720 &= 2^4 \times 3^2 \times 5 \\ 1080 &= 2^3 \times 3^3 \times 5 \\ \hline \text{H.C.F.} &= 2^3 \times 3^2 \times 5 \\ &= 360. \end{aligned}$$

Here 2 is a common factor three times, 3 is common twice, and 5 is common once.

The H.C.F. of two or more numbers must be the continued product of all the common prime factors of the numbers.

FURTHER ILLUSTRATIONS.

Ex. 1.

$$\begin{aligned} 792 &= 2^3 \times 3^2 \times 11 \\ 4368 &= \underline{2^4 \times 3 \times 7 \times 13} \\ \text{H.C.F.} &= 2^3 \times 3 \\ &= 24. \end{aligned}$$

Ex. 2.

$$\begin{aligned} 2730 &= 2 \times 3 \times 5 \times 7 \times 13 \\ 5304 &= 2^3 \times 3 \times 13 \times 17 \\ 780 &= \underline{2^2 \times 3 \times 5 \times 13} \\ \text{H.C.F.} &= 2 \times 3 \times 13 \\ &= 78. \end{aligned}$$

EXAMPLES XX.

Oral Exercises.

Find the H.C.F. of

1. 12 and 18.
2. 20 and 25.
3. 30 and 42.
4. 18 and 30.
5. 60 and 84.
6. 54 and 90.

Written Exercises.

7. 45 and 105.
8. 72 and 90.
9. 126 and 315.
10. 189 and 273.
11. 132 and 252.
12. 315 and 357.
13. 693 and 819.
14. 792 and 924.
15. 891 and 1221.
16. 48, 60, and 72.
17. 72, 108, and 180.
18. 264, 360, and 600.
19. 630, 756, and 1155.

92. We must now show how to find the H.C.F. of two numbers without going through the troublesome process of expressing the numbers as the product of prime factors.

The method depends on the following theorem, proved in Art. 77:

Any common factor of two numbers is also a factor of the sum, or of the difference, of any multiples of the numbers.

Suppose that we have two numbers whose H.C.F. is required.

If we divide the greater number by the smaller, then, by the nature of division,

(i) the remainder is equal to the difference between the greater number and some multiple of the smaller;

(ii) the greater number is equal to the sum of the remainder and some multiple of the smaller.

From (i) it follows that **any common factor of the original numbers** is a factor of the remainder, and therefore **is a common factor of the remainder and the smaller number.**

From (ii) it follows that **any common factor of the remainder and the smaller number** is a factor of the greater number also, and therefore **is a common factor of the two original numbers.**

The H.C.F. of the two original numbers must therefore be the same as the H.C.F. of the smaller number and the remainder.

Thus the problem of finding the H.C.F. of the two original numbers is reduced to that of finding the H.C.F. of the smaller number and the remainder.

Ex. 1. *Find the H.C.F. of* 3663 *and* 5439.

Divide the greater by the less.

$$\begin{array}{r} 3663)5439(1 \\ \underline{3663} \\ 1776 \end{array}$$

Hence the H.C.F. required is the same as the H.C.F. of 1776 and 3663. Divide the greater of these by the less.

$$\begin{array}{r} 1776)3663(2 \\ \underline{3552} \\ 111 \end{array}$$

The problem is now reduced to finding the H.C.F. of 111 and 1776. Again divide.

```
111)1776(16
    111
    ---
     666
     666
     ---
```

Thus, 111 is a factor of 1776, and therefore 111 is the H.C.F. of 111 and 1776.

But the H.C.F. of 111 and 1776 is the H.C.F. required.

The successive divisions are usually written in a more compact form, as follows:

```
3663)5439(1
     3663
     ----
     1776)3663(2
          3552
          ----
           111)1776(16
               111
               ---
                666
                666
                ---
```

Ex. 2. *Find the H.C.F. of* 311 *and* 331.

```
311)331(1
    311
    ---
     20)311(15
        20
        ---
        111
        100
        ---
         11)20(1
            11
            --
             9)11(1
                9
                -
                2)9(4
                  8
                  -
                  1)2(2
                    2
                    -
```

Here the H.C.F. of 311 and 331 is the same as the H.C.F. of 1 and 2, so that the numbers are prime to one another.

In this example, it would be a great waste of time to proceed to the end; for the H.C.F. required is the H.C.F. of *any* divisor and the corresponding dividend, and as soon as it is obvious that one such pair have no common factors it is not necessary to proceed further. Now the only prime factors of 20 are 2 and 5, and by inspection neither of these is a factor of 311.

93. The H.C.F. of numbers containing decimals is not often needed. The process, however, for finding such is as follows:

Arrange all the numbers, by annexing naughts, so that all shall have the same number of decimal places; then proceed as before.

Ex. 1. *Find the H.C.F. of* 1.08 *and* .072.

$$\begin{aligned} 1.080 &= 2^3 \times 3^3 \times 5 \times .001 \\ .072 &= 2^3 \times 3^2 \qquad \times .001 \\ \text{H.C.F.} &= 2^3 \times 3^2 \qquad \times .001 \\ &= .072. \end{aligned}$$

Ex. 2. *Find the H.C.F. of* .108 *and* .072.

$$\begin{aligned} .108 &= 2^2 \times 3^3 \times .001 \\ .072 &= 2^3 \times 3^2 \times .001 \\ \text{H.C.F.} &= 2^2 \times 3^2 \times .001 \\ &= .036. \end{aligned}$$

Ex. 3. *Find the H.C.F. of* 366.3 *and* 54.39.

$$\begin{aligned} 366.30 &= 2 \times 3^2 \times 5 \times 11 \times 37 \times .01 \\ 54.39 &= \quad 3 \times 7^2 \qquad \times 37 \times .01 \\ \text{H.C.F.} &= \quad 3 \times 37 \times .01 \\ &= 1.11. \end{aligned}$$

EXAMPLES XXI.

Written Exercises.

Find the H.C.F. (or G.C.M.) of

1. 221 and 247.
2. 357 and 391.
3. 899 and 1073.
4. 663 and 923.
5. 1517 and 1927.
6. 1785 and 2485.
7. 3499 and 3953.
8. 4899 and 5893.
9. 9709 and 22849.
10. 11663 and 12091.
11. 17947 and 29737.
12. 11453 and 12961.
13. 3834038 and 4169594.
14. 132038 and 369792.
15. 5411728 and 10902416.

To find the H.C.F. of three or more numbers, we have only to find the H.C.F. of the first two numbers; then the H.C.F. of this result and the third number; and so on.

Ex. *Find the H.C.F. of* 286, 338, *and* 585.

The H.C.F. of 286 and 338 is 26. Then the H.C.F. of 26 and 585 is 13.

Find the H.C.F. of

16. 165, 198, 242.

17. 312, 429, 572.

18. 222, 370, 550.

19. 2387, 2821, 4433.

20. 3157, 3321, 4059.

21. 4732, 5824, 6643.

22. What would be the answers to 7, 9, and 18, if they were as follows?

(7) 3.499 and 395.3.

(9) 97.09 and 2.2849.

(18) 2.22, 370, and 550.

LEAST COMMON MULTIPLE.

94. A number which is *exactly* divisible by two or more numbers is called a **Common Multiple** of those numbers.

For example, 200 is a C.M. of 20 and 25.

The smallest number which is *exactly* divisible by two or more numbers is called the **Least Common Multiple** (L.C.M.) of those numbers.

For example, 100 is the L.C.M. of 20 and 25.

95. When numbers are resolved into their *prime* factors, their *L.C.M.* can be found by inspection.

Consider, for example, the numbers 120, 252, and 3575.

$$\begin{aligned}
120 &= 2^3 \times 3 \times 5 \\
252 &= 2^2 \times 3^2 \times 7 \\
4125 &= \quad 3 \times 5^3 \times 11 \\
\hline
\text{L.C.M.} &= 2^3 \times 3^2 \times 5^3 \times 7 \times 11 \\
&= 693000.
\end{aligned}$$

Here we see that a common multiple must contain the prime factors 2, 3, 5, 7, and 11. Also a multiple of the first number must contain the *third* power at *least* of the factor 2; a multiple of the second must contain the *second* power at *least* of the factor 3; a multiple of the third number must contain the *third* power at *least* of the factor 5; and we must also have the first power at *least* of the factors 7 and 11.

The least common multiple must therefore be

$$2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 5 \times 7 \times 11.$$

The L.C.M. of two or more numbers must be the continued product of the highest powers of all the different prime factors of the numbers.

Illustration.

$$\begin{aligned} 105 &= 3 \times 5 \times 7 \\ 126 &= 2 \times 3^2 \times 7 \\ 196 &= 2^2 \times 7^2 \\ \hline \text{L.C.M.} &= 2^2 \times 3^2 \times 5 \times 7^2 \\ &= 8820. \end{aligned}$$

96. The form for finding the L.C.M. given in Art. 95 is excellent, because of the ease with which it may be applied to any example in which the H.C.F. or the L.C.M. is to be found, and because it involves more or less *mental* work, which is stimulating.

In obtaining the prime factors, the highest powers of the lowest primes should always be taken out first; thus, 2, 2^2, 2^3, etc., then 3, 3^2, 3^3, etc., should be the order in which the factors should be taken out.

The usual method by which the L.C.M. of numbers is found is as follows:

Having written the numbers in a row, first strike out any numbers which are factors of any of the others (for every multiple of a number is a multiple of any factor of that number). Then divide by any **prime** *which divides at least* two *of the numbers, and put the quotients below those numbers which can be divided, and bring down all those numbers which are not divisible by that prime. Operate on the second row in the same manner as on the first, and go on until a row is arrived at in which no two numbers have any*

common factor. Then the L.C.M. *is the continued product of all the divisors and all the numbers left in the last row.*

Thus, to find the L.C.M. of 2, 5, 15, 24, 25, 30, 36.

The process is written as follows:

$$\begin{array}{r}
2\,\underline{)\,\cancel{2},\ \cancel{5},\ \cancel{15},\ 24,\ 25,\ 30,\ 36} \\
2\,\underline{)\,12,\ 25,\ 15,\ 18} \\
3\,\underline{)\ \ 6,\ 25,\ 15,\ \ 9} \\
2,\ 25,\ \ \cancel{5},\ \ 3
\end{array}$$

Hence, the L.C.M. is $2 \times 2 \times 3 \times 2 \times 25 \times 3 = 1800$.

The reason the L.C.M. is given by the above process is that the L.C.M. of the numbers in the first row $= 2 \times$ L.C.M. of the numbers in the second row; and so on to the end.

EXAMPLES XXII.

Oral Exercises.

Find the L.M.C. of

1. 4, 10.
2. 6, 8.
3. 5, 10, 12.
4. 7, 6, 2.
5. 9, 12.
6. 3, 7, 10.
7. 12, 20.
8. 8, 12, 3.
9. 30, 50.
10. 10, 35.
11. 3, 4, 5, 6.
12. 2, 8, 12, 20.

Written Exercises.

Find the L.C.M. of

13. 12, 16.
14. 20, 25.
15. 25, 30.
16. 8, 12, 20.
17. 6, 8, 12.
18. 15, 20, 30.
19. 30, 45, 54.
20. 10, 14, 35.
21. 6, 12, 18, 63.
22. 9, 36, 45, 81.
23. 3, 11, 18, 33, 36.
24. 25, 27, 33, 55.
25. 16, 20, 22, 33, 36.
26. 3, 4, 10, 12, 14, 16, 18.
27. 3, 14, 12, 56, and 28.

28. 16, 18, 20, 24, 30, 36.

29. 12, 15, 18, 21, 25, 35, 210.

30. 12, 42, 49, 54.

31. 30, 35, 42, 60, 72.

32. 18, 54, 90, 102, 120, 144.

97. To find the L.C.M. of numbers which cannot be readily resolved into factors, we must first find the H.C.F.

For example, to find the L.C.M. of 4592 and 5371.

The H.C.F. is 41. And by division

$$4592 = 41 \times 112.$$
$$5371 = 41 \times 131.$$

Hence, as 112 and 131 are prime to one another, the L.C.M. is

$$41 \times 112 \times 131.$$

Hence the L.C.M. of two numbers equals one of the numbers multiplied by the quotient obtained by dividing the other number by their H.C.F.

It should be noticed that the L.C.M. $\times$ G.C.M. $= 41 \times 112 \times 41 \times 131 =$ product of the numbers.

To find the L.C.M of more than two numbers we can find the L.C.M. of two of the numbers, and then the L.C.M. of this result and of the third number, and so on to the end.

EXAMPLES XXIII.

Written Exercises.

Find the L.C.M. of

1. 357, 391.
2. 851, 943.
3. 3497, 4035.
4. 4899, 5893.
5. 165, 198, 242.
6. 312, 429, 572.
7. 360, 1350, 1500.
8. 195, 546, 286.
9. 420, 630, 1050, and 1470.
10. 1365, 2288, 2640.

98. By using the form of work as given in Art. 95, we may find the H.C.F. and the L.C.M. at the same time.

Ex. *Find the H.C.F. and the L.C.M. of* 24, 60, *and* 72.

$$\begin{aligned} 24 &= 2^3 \times 3 \\ 60 &= 2^2 \times 3 \times 5 \\ 72 &= 2^3 \times 3^2 \\ \hline \text{H.C.F.} &= 2^2 \times 3 \qquad = 12. \\ \text{L.C.M.} &= 2^3 \times 3^2 \times 5 = 360. \end{aligned}$$

It seems best to make a digression here to explain the use of the sign of *parenthesis,* and the use of *cancellation.*

PARENTHESIS.

99. The sign of parenthesis is made thus, ( ); in mathematics this sign is often called parenthesis.

100. Several numbers are included in parenthesis when they are to be *treated as a whole,* some sign being put outside the parenthesis to show *how* this whole is to be treated.

Thus, $17 + (4 + 8)$ denotes that 8 is to be added to 4 and that this result is then to be added to 17.

Again, $117 - 6 \times (4 + 8)$ denotes that 6 is to be multiplied by the sum of 4 and 8, and that the product is to be subtracted from 117.

Also, $36 \div (24 \div 4)$ denotes that 24 is to be divided by 4, and that 36 is to be divided by this result.

When two parentheses come together without any sign between them, the sign of multiplication must be understood.

Thus, $(5 + 7)(9 - 3)$ is put for $(5 + 7) \times (9 - 3)$.
Also, $6(5 + 7)$ is put for $6 \times (5 + 7)$.

101. Instead of enclosing numbers in parenthesis a line, called a vinculum, is sometimes drawn over them.

Thus, $17 + \overline{4 + 8}$ may be used instead of $17 + (4 + 8)$.

102. Sometimes a parenthesis is put within a parenthesis: to avoid confusion the parentheses are made of different shapes and are named as follows:

( ), sign of parenthesis,
{ }, braces,
[ ], brackets.

For example, $150 - 3\{13 - (9 - 3)\}$.

In order to simplify when there is more than one bracket, it will be found convenient to clear away the *innermost* bracket first.

Thus, $150 - 3\{13 - (9 - 3)\} = 150 - 3\{13 - 6\} = 150 - 3 \times 7$
$= 150 - 21 = 129$.

Operations of multiplication and division are to be performed in order from left to right, and each sign is a direction to multiply or divide by the number that follows next after it.

For example, $60 \div 6 \times 3 = 10 \times 3 = 30$;
and $60 \times 6 \div 3 = 360 \div 3 = 120$;
and $60 \div 10 \div 2 = 6 \div 2 = 3$.

EXAMPLES XXIV.

Written Exercises.

1. $8 + (7 + 3)$.
2. $15 - (9 + 3)$.
3. $27 - (11 - 4)$.
4. $15 - 2(8 - 5)$.
5. $3\,\overline{7 - 2} - 4(8 - 6)$.
6. $7(3 + 9) - 5(12 - 4)$.
7. $(.9 + .7)(.5 + 1.1)$.
8. $(13 - .5)(1.5 - .7)$.
9. $(23 - 12)(28 - 12)$.
10. $3(13 - 4)(15 - 7)$.
11. $8(1.7 - 1.1)(1.3 - .06)$.
12. $5(11 + 5)(11 - 5)$.
13. $325 - (17 - 2)(24 - 5)$.
14. $27 - 3(20 - 11) + 4(8 - 3)$.
15. $18 + \{17 - 2 - (15 - 4)\}$.
16. $23 - [41 - \{2 + 1\} - \overline{27 - 6}]$.

17. $(3.8016 - 2.794)(1.8093 - .078)$.

18. $12 \div 6 \times 2$.

19. $12 \div (6 \times 2)$.

20. $18 \times 6 \div 2$.

21. $18 \times (6 \div 2)$.

22. $28 \div 7 - 3 + 2$.

23. $28 \div (7 - 3) + 2$.

24. $28 \div [7 - (3 + 2)]$.

CANCELLATION.

103. Here we make use of the following principle:

Dividing both dividend and divisor by the same number does not change the quotient.

Thus, $24 \div 4 = 6$, and if both 24 and 4 be divided by 2, we shall have $12 \div 2 = 6$; also, if both 24 and 4 be divided by 10, we shall have $2.4 \div .4 = 6$.

104. NOTE. It should be noticed that although the *quotient* is unchanged by dividing both dividend and divisor by the same number, the *remainder*, if any, is not unchanged, but is equal to the original remainder divided by the number by which the original divisor and dividend were multiplied.

Therefore we must multiply the remainder, if any, by the number used in dividing, *if we wish* the remainder obtained by using the original numbers.

EX. 1. $44 \div 8 = 5$, *with a remainder of* 4;
while $11 \div 2 = 5$, " " " " 1.

It is evident that the remainder **1** compared with the divisor **2** is just as large as the remainder **4** compared with the divisor **8**.

EX. 2. *How many pieces, each* 24 *inches long, can be cut from a string* 231 *inches long, and what will be the length of the part left over?*

$231 \div 24 = 77 \div 8 = 9$, with a remainder of 5.

This remainder must be multiplied by 3, if we wish to know how many inches of string remain. *Ans.* = 9 pieces, with 15 inches remaining.

105. The principle of Art. 103 may be used in shortening the process of division, especially when the dividend and divisor can be factored at sight. Thus,

Ex. 1. $99\overline{)3663}$ gives the same quotient as $11\overline{)407}$
$37 = Ans.$

Ex. 2. $36\overline{)4884} = 6\overline{)814}$
135, rem. 4.

$4884 \div 36 = 135$, rem. 24. [Compare Art. 68.]

106. Division has thus far been indicated by the sign $\div$; but division is often indicated by writing the dividend just above the divisor with a line between; thus,

$36 \div 8$ is sometimes written $\frac{36}{8}$,

and what is true of the first expression is true of the second. In either case dividing both 36 and 8 by the same number does not change the quotient; thus,

$$\left.\begin{array}{l} 36 \div 8 = 9 \div 2 = 4, \text{ with a remainder of } 1. \\ \frac{36}{8} = \frac{9}{2} \qquad = 4, \quad \text{“} \quad \text{“} \qquad \text{“} \qquad \text{“ } 1. \end{array}\right\} \text{Art. } 104.$$

107. The process of dividing both dividend and divisor (or their factors) by the same factor is called **Cancellation.**

A thin line drawn through a dividend or divisor indicates that a factor has been cancelled; thus,

$$\frac{\overset{20}{\cancel{120}}}{\underset{7}{\cancel{42}}} = \frac{20}{7} = 2, \text{ with a remainder of } 6.$$

Here 6 is cancelled from dividend and divisor, and the quotients are written, one above and one below.

Ex. $(5 \times 48 \times 28) \div (10 \times 14 \times 9)$ may be written

$$\frac{5 \times 48 \times 28}{10 \times 14 \times 9}.$$

The latter is the better form when we wish to cancel; thus,

$$\frac{\overset{1}{\cancel{5}} \times \overset{16}{\cancel{48}} \times \overset{\overset{2}{\cancel{4}}}{\cancel{28}}}{\underset{\cancel{2}}{\cancel{10}} \times \underset{\cancel{2}}{\cancel{14}} \times \underset{3}{\cancel{9}}} = \frac{16}{3} = 5, \text{ with a remainder of } 1.$$

Here 5 is cancelled from 5 and 10, 3 from 48 and 9, 7 from 28 and 14, and the two 2's thus obtained in the divisor cancel with the 4 in the dividend. The result is the same as if there had been no cancellation.

EXAMPLES XXV.

Written Exercises.

Simplify by cancellation:

1. $\frac{27}{33}$.
4. $\frac{81}{99}$.
7. $\frac{77}{132}$.
10. $\frac{444}{468}$.
2. $\frac{12}{54}$.
5. $\frac{108}{144}$.
8. $\frac{187}{209}$.
11. $\frac{216}{306}$.
3. $\frac{18}{63}$.
6. $\frac{117}{144}$.
9. $\frac{264}{319}$.
12. $\frac{399}{703}$.

13. $\dfrac{7 \times 22}{11 \times 63}$.

14. $(8 \times 38 \times 41) \div (19 \times 4)$.

15. $\dfrac{6(42 - 3)}{13 \times 8}$.

16. $\dfrac{1.26 \times 3.5}{.6 \times 7}$.

17. Why can we not cancel in $\dfrac{12 + 13}{6 - 3}$?

EXAMPLES XXVI.

Miscellaneous Examples, Chap. III.

1. Divide the product of 8978 and 55112 by 5561.

2. Divide 210 dollars between A and B, so that A may have 5 times as much as B.

3. Find the H.C.F. of 3465 and 3696.

4. Multiply 606.78 by 11.

5. Find 35^2; 105^2; 7.5^2; $\sqrt{247009}$.

6. What number is the same multiple of 7 that 21560 is of 55?

7. What is the price of a silver bowl weighing 50 ounces, at 1.25 dollars an ounce?

8. Two equal sums were respectively divided among 12 men and a certain number of boys. Each man received 5 dollars, and each boy 1 dollar. How much was divided altogether, and how many boys were there?

9. Exactly 20 years ago, a man was four times as old as his son, whose present age is 28. What is the present age of the father?

10. Find 19×16; 656×125.

---

11. A certain chapter of a book begins at the top of the 357th page and ends at the bottom of the 435th page. How many pages are there in the chapter?

12. After multiplying 375 by 29, and 131 by some other number, the results when added amounted to 13888. What was the other number?

13. Find the H.C.F. of 5610, 11781, and 1309.

14. Find the least number by which 222 must be multiplied in order that the product may be a multiple of 1295.

15. Four bells toll at intervals of 3, 4, 5, and 6 seconds, respectively. If they all begin to toll at the same instant, how long will it be before they again all toll together?

---

16. Add fourteen hundred seventeen, four thousand eleven hundred nine, six million fifteen thousand, and eighteen million twelve hundred nineteen.

17. A certain number was divided by 35 by 'short' divisions; the quotient was 72, the first remainder was 2, and the second remainder was 6. What was the dividend?

18. Multiply 700630.0003 by 1006.07, and prove by dividing the product by the multiplier.

19. Find the continued product of 18, 13, and 11; obtain the square root of the product to two decimal places.

20. Divide 126819 by 21, using factors.

---

21. What is the least number of times that 315 must be added to 1594 that the sum may exceed a million?

22. Multiply 67412 by 9997 as shortly as you can.

23. Divide 789 by .10063 to 3 decimal places.

24. Find the H.C.F. of 10481 and 17617.

25. Four men can walk 30, 35, 40, and 45 miles a day, respectively; what is the least distance they can all walk in an exact number of days?

---

26. Find the L.C.M. of 12, 64, 80, 96, 120, 160.

27. Find the prime factors of 1176 and 19404, and hence write down their G.C.M. and L.C.M.

28. The quotient is twice the divisor, and the remainder which is 50 is one-fifth part of the quotient. Find the dividend.

29. Simplify $\frac{80 \times 51}{125 \times 219}$; obtain the answer in two forms.

30. Find the least number which can be divided by 7, 20, 28, and 35, and leave 3 as remainder in each case.

---

31. What number is that which after being subtracted 19 times from 1000 leaves a remainder of 12?

32. Multiply three thousand eighty-seven by seventy-two thousand nine hundred thirty. What numbers less than 12 will exactly divide the product?

33. (*a*) Simplify $650 \times 1.25 \div .5$.

(*b*) The answer is a multiple of which of the following numbers: 5, 15, 25, 65, 105, 125?

Obtain (*b*) by first obtaining primes of the answer.

34. Find $19 \times 17 \times 11 \times 2.5 \times 1.25$.

35. Find $65^2 \times .11$.

36. Simplify (*a*) $\frac{72}{9} \times 4 - 2 + 6\,(18 - 14)$.

(*b*) $\frac{72}{9} \times (4 - 2) + 6\,(18 - 14)$.

(*c*) $2(\frac{72}{9} \times 4) - \{2 + 6\,(18 - 14)\}$.

---

37. If a number when divided by 35 give a remainder 27, what remainder will it give when divided by 7?

38. What is the greatest and what is the least number of four digits which is exactly divisible by 73?

39. Find the H.C.F. and the L.C.M. of 21, 22, 24, 28, 32, 33; also of 16, 18, 20, 24, 30, 36.

40. Find the number nearest to 1000 and exactly divisible by 39.

41. Multiply 7863 by 999, and see if the product is divisible by 3.

42. Find $\sqrt{4912.888464}$.

43. Find $\sqrt{3^6}$.

(*a*) Divide the following numbers by 2.

(*b*) Prove your answers by first simplifying the numbers, and then dividing by 2.

44. $3\,(6 + 8)$; $3(6 \times 8)$.

45. $4\,(6 + 8)$; $4(6 \times 8)$.

46. $4\,(18 + 6)$; $4(18 \div 6)$.

47. $(6 \times 2)\,(8 + 10)$.

48. $6\,(12 \div 3) + 8\,\overline{6 + 4 + 2}$.

49. $28 \div [7 - (3 + 2)]$.

CHAPTER IV.

FRACTIONS.

108. If a unit be divided into 2, 3, 4, 5, etc., equal parts, these parts are called halves, third-parts, fourth-parts, fifth-parts, etc., or more shortly and more generally, *halves, thirds, fourths, fifths,* etc.

If the unit quantity be divided into any number of equal parts, one or more of these parts is called a **Fraction** of the unit.

For example, if a unit quantity, as one apple, be divided into sevenths, *three* of these parts constitute *three sevenths*, and the three sevenths is a *fraction* of seven sevenths, the unit quantity.

109. The number which indicates how many parts of the unit quantity are to be used is called the **Numerator.**

The number which indicates into how many parts the unit quantity is divided is called the **Denominator.**

110. The expression formed by writing a numerator just above a denominator with a line between is called a **Common Fraction.**

Thus, $\frac{3}{7}$, $\frac{8}{13}$ (eight-thirteenths), $\frac{1}{23}$ (one twenty-third), are common fractions (called briefly fractions).

Common fractions are also called *vulgar* fractions.

NOTE. A fraction is an expression of division, the numerator and denominator corresponding to the dividend and divisor respectively. What is true of dividend and divisor is true of numerator and denominator. When the indicated division is performed, the quotient is generally a decimal.

Ex. $\frac{3}{24} = 3 \div 24$.

$$\begin{array}{r@{}l} 24)3.000(.125 \\ \underline{2\,4} \\ 60 \\ \underline{48} \\ 120 \\ \underline{120} \end{array}$$

111. If we have 3 units, and divide each of them into 5 equal parts, and then take one of the parts from each divided unit, we shall take one part out of every five, that is, one-fifth of the whole three units; but each of the parts is one-fifth of a single unit and we take 3 of them: we therefore take 3 *fifths* of one unit.

Thus, 3 *fifths of* 1 *unit is the same as* 1 *fifth of* 3 *units.*

Hence, $\frac{3}{5}$, which by definition denotes 3 *fifths* of 1 unit, may also be considered to stand for 1 *fifth* of 3 units.

The same holds good for all other fractions; for example,

$\frac{3}{5}$ of 1 dollar $= \frac{1}{5}$ of 3 dollars;

and $\frac{7}{9}$ of 1 foot $= \frac{1}{9}$ of 7 feet.

EXAMPLES XXVII.

1. Write in figures the following fractions: five-ninths, six-elevenths, eleven twenty-thirds, sixteen twenty-sevenths, seventeen ninety-firsts, ninety-five one hundred fourths.

2. Write in words: $\frac{3}{7}$, $\frac{5}{8}$, $\frac{2}{11}$, $\frac{9}{23}$, $\frac{5}{31}$, $\frac{17}{56}$, $\frac{7}{81}$, $\frac{90}{111}$, $\frac{15}{219}$, $\frac{119}{2000}$.

112. The numerator and denominator of a fraction are called its **Terms.**

When the numerator is less than the denominator, the fraction is called a **Proper Fraction**; and when the numerator is equal to or greater than the denominator, the fraction is called an **Improper Fraction.**

A number made up of an integer and a fraction is called a **Mixed Number.**

Thus, $2\frac{1}{4}$ (2 and $\frac{1}{4}$), which means $2 + \frac{1}{4}$, is a mixed number.

Changing the form of an expression, or changing the units in terms of which any quantity is expressed, is called **Reduction.**

113. Reducing a mixed number to an improper fraction.

Consider, for example, $3\frac{2}{7}$.

Each unit contains 7 *sevenths*, therefore 3 units contain 3 times 7 *sevenths*.

Hence, $3\frac{2}{7} = 3$ times 7 *sevenths* + 2 *sevenths* $= \dfrac{21+2}{7} = \dfrac{23}{7}$.

Again, $$7\tfrac{2}{9} = \frac{7 \times 9 + 2}{9} = \frac{65}{9}.$$

From the above it will be seen that a *mixed number is equivalent to an improper fraction* whose denominator is the denominator of the fractional part, and whose numerator is obtained by multiplying the integral part by the denominator of the fraction and adding its numerator.

It should be noticed that a whole number can be expressed as a fraction with *any* given denominator. For example,

$6 = 6 \times 7$ *sevenths* $= \frac{42}{7}$; also, $6 = 6 \times 13$ *thirteenths* $= \frac{78}{13}$.

114. Conversely, **reducing an improper fraction to a whole or mixed number.**

Consider, for example, $\frac{23}{7}$.

Since 7 *sevenths* make 1 unit,

$\frac{23}{7} = 3 \times 7$ *sevenths* + 2 *sevenths* $= 3 + 2$ *sevenths* $= 3\frac{2}{7}$.

Again, $\frac{24}{4} = 6 \times 4$ *fourths* $= 6$, since 4 *fourths* $= 1$.

From the above it will be seen that an *improper fraction is reduced to a mixed number* by dividing its numerator by its denominator; the quotient will form the integral part, while the remainder, if any, will form the numerator of the fractional part, whose denominator must be the denominator of the improper fraction.

EXAMPLES XXVIII.

Oral Exercises.

Express as improper fractions:

1. $1\frac{1}{2}$, $1\frac{3}{4}$, $2\frac{1}{3}$, $3\frac{1}{5}$. 2. $7\frac{3}{10}$, $6\frac{5}{12}$, $5\frac{2}{7}$, $3\frac{4}{9}$.

3. $4\frac{3}{11}$, $9\frac{2}{7}$, $12\frac{1}{5}$, $11\frac{6}{11}$.

4. Express 3, 5, and 9 as fractions with a denominator 7.

Express as whole or mixed numbers:

5. $\frac{12}{4}$, $\frac{20}{5}$, $\frac{35}{7}$, $\frac{49}{7}$. 6. $\frac{17}{11}$, $\frac{35}{9}$, $\frac{79}{12}$, $\frac{83}{10}$. 7. $\frac{10}{3}$, $\frac{17}{5}$, $\frac{25}{6}$, $\frac{50}{10}$.

Written Exercises.

Express as improper fractions:

8. $4\frac{3}{19}$, $5\frac{6}{17}$, $18\frac{6}{17}$. [Art. 55, I.]

9. $2\frac{112}{113}$, $7\frac{11}{212}$.

10. Reduce 13 to fifteenths, and 41 to twenty-fifths.

11. Express 427 as a fraction with a denominator 99.

Express as whole or mixed numbers:

12. $\frac{35}{17}$, $\frac{70}{29}$, $\frac{60}{21}$. 13. $\frac{422}{111}$, $\frac{516}{17}$, $\frac{357}{41}$. 14. $\frac{96}{21}$, $\frac{128}{42}$, $\frac{799}{56}$.

See Art. 70 for examples in 14.

115. Reducing a fraction to its lowest terms.

A fraction is said to be in its **Lowest Terms** when the numerator and denominator have no common factor.

Thus, the fractions, $\frac{2}{3}$, $\frac{5}{6}$, $\frac{12}{25}$, are in their lowest terms; but the fractions, $\frac{4}{6}$, $\frac{10}{12}$, $\frac{24}{50}$, are not in their lowest terms, for in each case the numerator and denominator have 2 as common factor.

The following is a very important truth:

The value of a fraction is not changed by dividing the numerator and denominator by the same number.

This truth is but a repetition of the principle stated in Art. 103.

Ex. *Reduce* $\frac{825}{1540}$ *to its lowest terms.*

To reduce to the *lowest* terms we must divide by the H.C.F. of the numerator and denominator; for we thus obtain an equivalent fraction whose numerator and denominator have no common factors.

In the present case the H.C.F. will be found to be 55.

Thus, $$\frac{825}{1540}=\frac{825 \div 55}{1540 \div 55}=\frac{15}{28}.$$

Instead of reducing a fraction to its lowest terms by dividing the numerator and denominator by their H.C.F., we may divide by *any* common factor, and repeat the process until the fraction is reduced to its lowest terms. Thus,

$$\frac{825}{1540}=\frac{165}{308}=\frac{15}{28}.$$

We see at once that 5 is a common factor; we therefore divide the numerator and denominator by 5, and obtain the equivalent fraction $\frac{165}{308}$. We now see that 11 is a common factor, and having divided the numerator and denominator by 11, we have the equivalent fraction $\frac{15}{28}$, which is at once seen to be in its lowest terms.

EXAMPLES XXIX.

Oral Exercises.

Reduce to their lowest terms:

1. $\frac{2}{4}, \frac{4}{6}, \frac{9}{12}, \frac{12}{16}, \frac{5}{15}$.
2. $\frac{15}{20}, \frac{12}{18}, \frac{8}{20}, \frac{15}{24}, \frac{24}{30}$.
3. $\frac{14}{21}, \frac{42}{54}, \frac{56}{77}, \frac{18}{30}$.
4. $\frac{24}{60}, \frac{25}{75}, \frac{56}{70}, \frac{70}{84}, \frac{64}{96}$.

Written Exercises.

Reduce to their lowest terms:

5. $\frac{144}{156}$.
6. $\frac{120}{135}$.
7. $\frac{120}{280}$.
8. $\frac{231}{273}$.
9. $\frac{792}{1683}$
10. $\frac{840}{3080}$.
11. $\frac{1001}{1155}$.
12. $\frac{1110}{1221}$.
13. $\frac{735}{1323}$.
14. $\frac{2028}{2340}$.
15. $\frac{3663}{4477}$.
16. $\frac{728}{2128}$.

116. Reducing fractions to equivalent fractions having the lowest common denominator.

The following is a very important truth:

The value of a fraction is not changed by multiplying the numerator and denominator by the same number.

This truth is but a repetition of the principle stated in Art. 67.

Consider the fractions, $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{8}{9}$.

The L.C.M. of the denominators 4, 6, and 9 is easily seen to be 36. Since 36 is a multiple of each denominator, all the fractions can be reduced to equivalent fractions which have 36 for denominator, provided the numerator and denominator of each of the fractions be multiplied by a suitable number, namely, by the numbers $36 \div 4$, $36 \div 6$, and $36 \div 9$, respectively; that is, by 9, 6, and 4, respectively.

Thus,
$$\frac{3}{4} = \frac{3 \times 9}{4 \times 9} = \frac{27}{36},$$
$$\frac{5}{6} = \frac{5 \times 6}{6 \times 6} = \frac{30}{36},$$
and
$$\frac{8}{9} = \frac{8 \times 4}{9 \times 4} = \frac{32}{36}.$$

Again, reduce $\frac{11}{18}$, $\frac{7}{30}$, $\frac{5}{24}$, to equivalent fractions having the lowest common denominator.

FULL WORK ILLUSTRATED.

$$\begin{aligned} 18 &= 2 \times 3^2 \\ 30 &= 2 \times 3 \times 5 \\ 24 &= 2^3 \times 3 \\ \hline \text{L.C.M} &= 2^3 \times 3^2 \times 5 \end{aligned}$$

$$\frac{11}{18} = \frac{11 \times 2^2 \times 5}{18 \times 2^2 \times 5} = \frac{220}{360},$$
$$\frac{7}{30} = \frac{7 \times 2^2 \times 3}{30 \times 2^2 \times 3} = \frac{84}{360},$$
$$\frac{5}{24} = \frac{5 \times 3 \times 5}{24 \times 3 \times 5} = \frac{75}{360}.$$

117. Comparison of Fractions.

Of two fractions which have the same denominator, the greater is that which has the greater numerator; for, the parts being the same, the greater fraction is that which has the most of them.

Again, of two fractions which have the same numerator, the greater is that which has the smaller denominator; for, the number of parts being the same in both, the greater is that in which the parts are the greater; that is, in which the unit has been divided into the smaller number of equal parts.

We can therefore see at once which of a number of fractions is the greatest, and which is the least, provided the fractions are first of all reduced to equivalent fractions with the same denominator. For this particular purpose it would do equally well to reduce the fractions to equivalent fractions with the same *numerator*, but it is for other purposes much less convenient to reduce fractions to equivalent fractions with the same numerator.

Ex. *Which is the greatest and which is the least of the fractions,* $\frac{3}{4}$, $\frac{5}{6}$, $\frac{8}{9}$?

As in the preceding article, the fractions are equivalent to $\frac{27}{36}$, $\frac{30}{36}$, and $\frac{32}{36}$, respectively; they are therefore in ascending order of magnitude.

EXAMPLES XXX.

Written Exercises.

Reduce to equivalent fractions with the lowest common denominator, and arrange in ascending order of magnitude:

1. $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}$.
2. $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}$.
3. $\frac{1}{5}, \frac{3}{10}, \frac{7}{20}$.
4. $\frac{5}{6}, \frac{7}{12}, \frac{11}{18}$.
5. $\frac{4}{9}, \frac{7}{15}, \frac{9}{40}$.
6. $\frac{5}{14}, \frac{11}{21}, \frac{24}{35}$.
7. $\frac{7}{8}, \frac{6}{7}, \frac{4}{5}$.
8. $\frac{7}{12}, \frac{11}{16}, \frac{13}{20}$.
9. $\frac{2}{3}, \frac{3}{5}, \frac{5}{7}$.
10. $\frac{5}{6}, \frac{8}{9}, \frac{9}{11}$.
11. $\frac{11}{16}, \frac{13}{18}, \frac{19}{24}$.
12. $\frac{22}{27}, \frac{19}{24}, \frac{29}{36}$.
13. $\frac{8}{15}, \frac{23}{45}, \frac{31}{60}$.
14. $\frac{5}{7}, \frac{2}{9}, \frac{1}{2}, \frac{9}{14}, \frac{8}{15}$.
15. $\frac{5}{24}, \frac{11}{30}, \frac{21}{40}, \frac{29}{48}$.
16. $\frac{4}{7}, \frac{7}{10}, \frac{7}{12}, \frac{19}{35}$.
17. $\frac{3}{8}, \frac{5}{12}, \frac{7}{18}, \frac{11}{30}$.
18. $\frac{6}{7}, \frac{7}{12}, \frac{17}{18}, \frac{19}{21}, \frac{25}{28}$.

Reduce to equivalent fractions which have the lowest common numerator:

19. $\frac{5}{8}, \frac{10}{17}, \frac{15}{23}$.
20. $\frac{12}{17}, \frac{16}{19}, \frac{18}{23}$.
21. $\frac{15}{37}, \frac{18}{43}, \frac{20}{49}$.

118. **Addition of Fractions.**

Fractions which have the same denominator are called **Similar Fractions.**

If fractions are dissimilar they must be made similar [Art. 116]; then their numerators may be added, and the sum written as a numerator for the common denominator. [Compare Art. 32.]

Ex. 1. *Find* $\frac{5}{6}+\frac{7}{12}+\frac{4}{9}$.

$$\frac{5}{6}=\frac{5\times 6}{6\times 6}=\frac{30}{36}$$
$$\frac{7}{12}=\frac{7\times 3}{12\times 3}=\frac{21}{36}$$
$$\frac{4}{9}=\frac{4\times 4}{9\times 4}=\frac{16}{36}$$
$$\text{Sum}=\frac{67}{36}=1\tfrac{31}{36}.$$

Or,

$$\begin{array}{r|l} & 36 \\ \hline \frac{5}{6}= & 30 \\ \frac{7}{12}= & 21 \\ \frac{4}{9}= & 16 \\ \hline \end{array}$$
$$\text{Sum}=\frac{67}{36}=1\tfrac{31}{36}.$$

After a little practice the middle column might be omitted.

Ex. 2. *Find* $2\frac{5}{6}+3\frac{7}{12}$.

$$2\tfrac{5}{6}=2\tfrac{10}{12}$$
$$3\tfrac{7}{12}=3\tfrac{7}{12}$$
$$\text{Sum}=6\tfrac{5}{12}$$

Here the 12ths are added, and 1 is carried to units. The process is similar to that represented in Ex. 2, Art. 29.

The result should in all cases be reduced to its lowest terms, and an improper fraction should be expressed as a mixed number.

EXAMPLES XXXI.

Oral Exercises.

Find the sum of the following fractions:

1. $\frac{1}{4}$ and $\frac{3}{4}$.
2. $\frac{3}{8}$ and $\frac{1}{8}$.
3. $\frac{5}{12}$ and $\frac{3}{12}$.
4. $\frac{7}{11}$ and $\frac{6}{11}$.
5. $\frac{5}{9}$ and $\frac{8}{9}$.
6. $\frac{11}{16}$ and $\frac{13}{16}$.
7. $\frac{1}{2}$ and $\frac{3}{4}$.
8. $\frac{3}{4}$ and $\frac{5}{6}$.
9. $\frac{3}{4}$ and $\frac{1}{8}$.

10. $\frac{5}{6}$ and $\frac{7}{8}$.

11. $\frac{2}{3}$ and $\frac{4}{9}$.

12. $2\frac{1}{5}$ and $3\frac{3}{5}$.

13. $4\frac{7}{11}$ and $6\frac{9}{11}$.

14. $20\frac{9}{16}$ and $10\frac{5}{16}$.

15. $3\frac{1}{6}$ and $4\frac{2}{3}$.

16. $8\frac{5}{9}$ and $6\frac{5}{18}$.

17. $12\frac{8}{11}$ and $6\frac{9}{22}$.

18. $4\frac{5}{7}$ and $6\frac{7}{42}$.

Written Exercises.

Find the sum of

19. $\frac{5}{16}$ and $\frac{7}{20}$.

20. $\frac{4}{15}$ and $\frac{7}{18}$.

21. $\frac{5}{21}$ and $\frac{9}{28}$.

22. $2\frac{3}{5}$ and $1\frac{7}{10}$.

23. $5\frac{1}{8}$ and $2\frac{7}{12}$.

24. $7\frac{5}{6}$ and $\frac{4}{9}$.

25. $2\frac{5}{6}$ and $3\frac{8}{9}$.

26. $3\frac{2}{3}$ and $1\frac{1}{4}$.

27. $1\frac{22}{27}$ and $7\frac{19}{24}$.

28. $\frac{3}{5}$, $\frac{4}{9}$, $\frac{7}{15}$, and $\frac{13}{45}$.

29. $\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, and $\frac{2}{9}$.

30. $\frac{3}{16}$, $\frac{5}{16}$, $\frac{7}{16}$, $\frac{9}{16}$, $\frac{11}{16}$, and $\frac{13}{16}$.

31. $\frac{5}{24}$, $\frac{7}{24}$, $\frac{11}{24}$, $\frac{13}{24}$, $\frac{17}{24}$, and $\frac{23}{24}$.

32. Find $\frac{2}{7} + \frac{3}{14} + \frac{4}{21} + \frac{5}{28} + \frac{6}{35}$.

33. Find $\frac{4}{9} + \frac{5}{18} + \frac{7}{27} + \frac{8}{45} + \frac{11}{60}$.

34. Find $3\frac{3}{5} + \frac{7}{12} + 5\frac{3}{16} + 7\frac{4}{15}$.

35. Find $3\frac{1}{4} + \frac{3}{10} + 5\frac{1}{35} + 3\frac{5}{28} + 7\frac{20}{63}$.

36. Find $3\frac{1}{6} + 5\frac{3}{8} + \frac{11}{24} + 3\frac{7}{30}$.

37. Find $\frac{11}{30} + 2\frac{7}{45} + 11\frac{3}{40} + 5\frac{13}{60}$.

38. Find $3\frac{3}{7} + 7\frac{7}{10} + \frac{5}{12} + 2\frac{19}{35}$.

39. Find $2\frac{13}{357} + 5\frac{27}{391}$.

40. Find $3\frac{56}{165} + 5\frac{49}{198} + 11\frac{105}{242}$.

41. Find $10\frac{121}{312} + 11\frac{98}{429} + 7\frac{255}{572}$.

119. Subtraction of Fractions.

Subtraction can be performed with fractions only when they are similiar. [Compare Art. 39.]

Ex. 1. *Subtract* $\frac{5}{8}$ *from* $\frac{11}{12}$.

$$\frac{11}{12} = \frac{22}{24}$$
$$\frac{5}{8} = \frac{15}{24}$$
$$\text{Difference} = \frac{7}{24}.$$

Ex. 2. *Find* $5\frac{11}{12} - 3\frac{5}{8}$.

$$5\frac{11}{12} = 5\frac{22}{24}$$
$$3\frac{5}{8} = 3\frac{15}{24}$$
$$\text{Remainder} = 2\frac{7}{24}.$$

Ex. 3. *Subtract* $3\frac{11}{12}$ *from* $5\frac{5}{8}$.

$$5\frac{5}{8} = 5\frac{15}{24}$$
$$3\frac{11}{12} = 3\frac{22}{24}$$
$$\text{Remainder} = 1\frac{17}{24}.$$

Here $\frac{22}{24}$ cannot be subtracted from $\frac{15}{24}$, therefore we take 1 unit from the 5 units and add it (changed to 24ths) to $\frac{15}{24}$, making $\frac{39}{24}$; now $\frac{22}{24}$ from $\frac{39}{24}$ equals $\frac{17}{24}$, and 3 from 4 equals 1.

The operation is similar to that represented in Ex. 2, Art. 38.

Ex. 4. *Simplify* $3\frac{1}{8} - 2\frac{5}{6} + 8\frac{3}{4} - 5\frac{7}{12} - 2\frac{7}{24} + \frac{17}{48}$. [See Art. 41.]

$$\begin{array}{rlrl} 3\frac{1}{8} = & 3\frac{6}{48} & 2\frac{5}{6} = & 2\frac{40}{48} \\ 8\frac{3}{4} = & 8\frac{36}{48} & 5\frac{7}{12} = & 5\frac{28}{48} \\ \frac{17}{48} = & \frac{17}{48} & 2\frac{7}{24} = & 2\frac{14}{48} \\ \hline & 12\frac{11}{48} & - & 10\frac{34}{48} = 1\frac{25}{48}. \end{array}$$

EXAMPLES XXXII.

Oral Exercises.

Simplify (give lowest terms in your answers):

1. $\frac{7}{12} - \frac{5}{12}$.	5. $\frac{3}{4} - \frac{1}{2}$.	9. $\frac{5}{12} - \frac{2}{7}$.
2. $\frac{5}{8} - \frac{3}{8}$.	6. $\frac{5}{6} - \frac{3}{4}$.	10. $3\frac{9}{11} - 1\frac{7}{11}$.
3. $\frac{9}{16} - \frac{5}{16}$.	7. $\frac{3}{5} - \frac{4}{7}$.	11. $6\frac{1}{6} - 4\frac{2}{3}$.
4. $\frac{17}{32} - \frac{9}{32}$.	8. $\frac{7}{12} - \frac{3}{8}$.	12. $5\frac{5}{7} - 2\frac{7}{42}$.

Written Exercises.

Simplify;

13. $\frac{13}{16} - \frac{7}{10}$.	15. $\frac{7}{18} - \frac{13}{54}$.	17. $\frac{7}{9} - \frac{3}{16}$.
14. $\frac{3}{14} - \frac{2}{21}$.	16. $\frac{5}{16} - \frac{7}{24}$.	18. $\frac{5}{12} - \frac{2}{7}$.

19. $\frac{8}{15}-\frac{7}{20}$.

20. $\frac{8}{65}-\frac{3}{26}$.

21. $\frac{9}{68}-\frac{11}{85}$.

22. $\frac{6}{19}-\frac{21}{76}$.

23. $3\frac{3}{4}-2\frac{1}{2}$.

24. $7\frac{19}{21}-5\frac{11}{42}$.

25. $2\frac{35}{36}-2\frac{19}{30}$.

26. $3\frac{5}{6}-2\frac{7}{18}$.

27. $7\frac{8}{15}-5\frac{3}{10}$.

28. $19\frac{11}{21}-12\frac{5}{14}$.

29. $7\frac{3}{14}-6\frac{9}{77}$.

30. $3\frac{7}{45}-2\frac{3}{25}$.

31. $6\frac{7}{8}-2\frac{15}{16}$.

32. $5\frac{4}{15}-3\frac{7}{20}$.

33. $16\frac{7}{30}-5\frac{11}{36}$.

34. $9\frac{7}{63}-4\frac{98}{105}$.

Find the difference between

35. $3\frac{9}{11}$ and $5\frac{6}{17}$.

36. $7\frac{8}{21}$ and $8\frac{7}{24}$.

37. $6\frac{20}{77}$ and $15\frac{25}{99}$.

38. $7\frac{11}{42}$ and $5\frac{25}{63}$.

39. $8\frac{21}{40}$ and $12\frac{17}{48}$.

40. $6\frac{21}{52}$ and $12\frac{17}{39}$.

41. $143\frac{9}{65}$ and $127\frac{11}{117}$.

42. $85\frac{60}{697}$ and $72\frac{81}{615}$.

Simplify:

43. $2\frac{1}{2}+3\frac{1}{4}-4\frac{1}{8}$.

44. $6\frac{11}{12}-2\frac{7}{8}+1\frac{5}{16}$.

45. $5\frac{3}{4}-3\frac{7}{16}+\frac{17}{20}-2\frac{7}{10}$.

46. $15\frac{7}{27}-13\frac{5}{12}+16\frac{7}{18}-9\frac{25}{54}$.

47. $12\frac{5}{42}-10+7\frac{10}{21}-\frac{4}{9}-5\frac{27}{35}$.

48. $4\frac{3}{8}-2\frac{7}{18}+2\frac{13}{28}-3\frac{5}{36}$.

49. $6\frac{4}{25}-\frac{3}{8}-2\frac{7}{40}+5\frac{2}{125}+\frac{3}{1000}$.

120. Multiplication by a Whole Number.

Fractions may be treated as concrete numbers; therefore,

as 3 times 5 tons $= 15$ tons,

so 3 " 5 sevenths $= 15$ sevenths;

i.e., 3 " $\frac{5}{7} = \frac{15}{7}$.

Again, 3 " $\frac{5}{18} = \frac{15}{18} = \frac{5}{6}$ (by cancellation)

i.e., 3 " $\frac{5}{18} = \frac{5}{18 \div 3} = \frac{5}{6}$.

Hence, to multiply a fraction by a whole number, we must multiply the numerator, or (when possible) divide the denominator, by that whole number.

The product should always be reduced to its lowest terms, and an improper fraction should be expressed as a mixed number.

Ex. *Multiply* $\frac{5}{18}$ *by* 15.

$$\frac{5}{18}\times 15=\frac{5\times 15}{18}=\frac{75}{18}=\frac{25}{6}=4\tfrac{1}{6}.$$

EXAMPLES XXXIII.

Oral Exercises.

Multiply and reduce to their simplest forms:

1. $\frac{2}{7}$ by 2.
2. $\frac{5}{19}$ by 3.
3. $\frac{4}{21}$ by 3.
4. $\frac{3}{17}$ by 4.
5. $\frac{3}{7}$ by 4.
6. $\frac{7}{12}$ by 4.
7. $\frac{5}{9}$ by 3.
8. $\frac{16}{17}$ by 17.
9. $\frac{7}{11}$ by 6.
10. $\frac{7}{12}$ by 8.

Written Exercises.

Perform the following examples (see Art. 107):

11. $\frac{4}{15}\times 10$.
12. $\frac{5}{16}\times 8$.
13. $2\frac{3}{4}\times 6$.
14. $5\frac{3}{8}\times 10$.
15. $7\frac{8}{15}\times 25$.
16. $\frac{49}{90}\times 15$.
17. $9\frac{41}{50}\times 25$.
18. $\frac{18}{13}\times 16$. [Art. 55.]
19. $\frac{116}{33}\times 22$.
20. $44\frac{2}{13}\times 26$.
21. $99\times\frac{642}{27}$.
22. $\frac{35}{4968}\times 9$.

121. Multiplication by a Fraction.

We understand multiplication to be the taking one number as many times as there are units in another. Thus, to multiply 5 by 4, we take as many fives as there are units in 4. Now 4 is $1+1+1+1$, and 5×4 is $5+5+5+5$.

Thus, *to multiply one number by a second is to do to the first what is done to the unit to obtain the second.*

For example, to multiply $\frac{5}{7}$ by $\frac{3}{4}$, we must do to $\frac{5}{7}$ what is done to the unit to obtain $\frac{3}{4}$; that is, we must divide $\frac{5}{7}$ into 4 equal parts

and take 3 of those parts. Each of the 4 parts into which $\frac{5}{7}$ is divided will be $\frac{5}{7 \times 4}$, and by taking 3 of these parts we get $\frac{5 \times 3}{7 \times 4}$.

Thus, $$\frac{5}{7} \times \frac{3}{4} = \frac{5 \times 3}{7 \times 4}.$$

Hence, the product of any two fractions is another fraction whose numerator is the product of their numerators and whose denominator is the product of their denominators.

The continued product of any number of fractions is obtained by continued application of the above rule.

Thus, to find the continued product of $\frac{2}{3}$, $\frac{4}{5}$, and $\frac{8}{9}$.

$$\frac{2}{3} \times \frac{4}{5} \times \frac{8}{9} = \frac{2 \times 4}{3 \times 5} \times \frac{8}{9} = \frac{2 \times 4 \times 8}{3 \times 5 \times 9} = \frac{64}{135}.$$

Hence, the product of any number of fractions is another fraction whose numerator is the product of their numerators and whose denominator is the product of their denominators.

It should be noticed that *the product of one fraction by a second is equal to the product of the second by the first.*

It should be noticed also that an *integer* $\times$ *a fraction equals* (*the integer* $\times$ *the numerator*) $\div$ *the denominator.*

Ex. 1. *Multiply* $\frac{6}{35}$ *by* $\frac{7}{27}$.

$$\frac{6}{35} \times \frac{7}{27} = \frac{\overset{2}{\cancel{6}} \times \overset{1}{\cancel{7}}}{\underset{5}{\cancel{35}} \times \underset{9}{\cancel{27}}} = \frac{2}{45}. \text{ [Art. 107.]}$$

Ex. 2. *Simplify* $\frac{2}{3} \times \frac{3}{4} \times \frac{2}{5}$.

$$\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{1}{\cancel{3}}}{\underset{\underset{1}{\cancel{2}}}{\cancel{4}}} \times \frac{\overset{1}{\cancel{2}}}{5} = \frac{1}{5}.$$

Ex. 3. *Multiply* $2\frac{1}{4}$ *by* $3\frac{5}{8}$.

The mixed numbers must first be reduced to improper fractions.

Thus, $$2\tfrac{1}{4} \times 3\tfrac{5}{8} = \frac{9}{4} \times \frac{29}{8} = \frac{9 \times 29}{4 \times 8} = \frac{261}{32} = 8\tfrac{5}{32}.$$

EXAMPLES XXXIV.

Simplify:

Oral Exercises.

1. $\frac{1}{3} \times \frac{1}{2}$.
2. $\frac{2}{3} \times \frac{5}{7}$.
3. $\frac{3}{5} \times \frac{3}{8}$.
4. $\frac{5}{7} \times \frac{4}{9}$.
5. $1\frac{1}{2} \times 2\frac{1}{4}$.
6. $2\frac{1}{4} \times 3\frac{2}{5}$.
7. $\frac{2}{3} \times \frac{3}{4}$.
8. $\frac{5}{9} \times \frac{12}{11}$.
9. $\frac{3}{5} \times \frac{20}{33}$.
10. $\frac{5}{6} \times \frac{12}{25}$.
11. $\frac{7}{11} \times \frac{22}{35}$.
12. $\frac{3}{7} \times \frac{21}{60}$.
13. $\frac{5}{9} \times \frac{27}{40}$.
14. $\frac{23}{40} \times \frac{40}{69}$.
15. $\frac{30}{81} \times \frac{9}{20}$.
16. $\frac{15}{16} \times \frac{8}{45}$.
17. $(\frac{2}{3})^2$.
18. $(\frac{4}{7})^2$.
19. $(\frac{2}{3})^3$.

Written Exercises.

20. $5\frac{2}{11} \times 3\frac{9}{19}$.
21. $\frac{2}{3} \times \frac{3}{4} \times \frac{4}{5}$.
22. $\frac{5}{8} \times \frac{2}{5} \times \frac{4}{7}$.
23. $\frac{3}{7} \times \frac{8}{11} \times \frac{7}{24}$.
24. $\frac{6}{7} \times 1\frac{3}{4} \times 5\frac{5}{6}$.
25. $2\frac{1}{4} \times 3\frac{1}{5} \times 4\frac{1}{6}$.
26. $5\frac{1}{5} \times 7\frac{1}{7} \times \frac{7}{10}$.
27. $\frac{3}{17} \times 5\frac{3}{13} \times \frac{13}{12}$.
28. $2\frac{1}{10} \times 3\frac{3}{7} \times 6\frac{1}{9}$.
29. $1\frac{7}{15} \times 2\frac{2}{9} \times 1\frac{7}{11}$.
30. $6\frac{9}{17} \times \frac{34}{65} \times 1\frac{28}{37}$.
31. $\frac{65}{114} \times 2\frac{11}{26} \times 5\frac{9}{11} \times \frac{19}{160}$.
32. $1\frac{1}{33} \times 1\frac{23}{40} \times \frac{20}{57} \times 1\frac{90}{119}$.
33. $\frac{39}{44} \times 1\frac{2}{5} \times 5\frac{7}{13} \times 2\frac{13}{21}$.
34. $(\frac{3}{5})^3$.
35. $(\frac{9}{20})^2$.
36. $(\frac{7}{11})^4$.

122. Division by a Whole Number.

Just as 15 *tons* ÷ 3 = 5 *tons*,
so also 15 *sevenths* ÷ 3 = 5 *sevenths*,
that is, $\frac{15}{7} \div 3 = \frac{5}{7}$.

Again, to divide $\frac{5}{6}$ by 3.

Here 5 is not a multiple of 3. But, since $\frac{5}{6} = \frac{5 \times 3}{6 \times 3}$,

$$\frac{5}{6} \div 3 = \frac{5 \times 3}{6 \times 3} \div 3 = \frac{5}{6 \times 3} = \frac{5}{18}.$$

We see at once, that a fraction is divided by a whole number by multiplying its denominator by the whole number. For example, in $\frac{5}{6 \times 3}$ there are the same number of parts as in $\frac{5}{6}$, namely five, but the unit in the former case is divided into 3 times as many parts as in the latter, and therefore each of the parts in $\frac{5}{6 \times 3}$ is one-third of each of the parts in $\frac{5}{6}$.

Hence, **to divide a fraction by a whole number, we must divide the numerator, or multiply the denominator (only when necessary), by that whole number.**

Ex. 1. *Divide* $3\frac{1}{5}$ *by* 7.

$$3\tfrac{1}{5} \div 7 = \frac{16}{5} \div 7 = \frac{16}{5 \times 7} = \frac{16}{35}.$$

Ex. 2. *Divide* $215\frac{3}{7}$ *by* 9.

$$\begin{aligned} 215\tfrac{3}{7} \div 9 &= 23 + 8\tfrac{3}{7} \div 9 \\ &= 23 + \tfrac{59}{7} \div 9 \\ &= 23\tfrac{59}{63}. \end{aligned}$$

When the integral part of the dividend is large, we first divide the integer by the divisor; then the remainder + the fractional part is to be divided by the divisor.

EXAMPLES XXXV.

Simplify:

Oral Exercises.

1. $\frac{4}{5} \div 2$.
2. $\frac{4}{5} \div 3$.
3. $\frac{8}{15} \div 4$.
4. $\frac{9}{25} \div 3$.
5. $\frac{6}{7} \div 8$.
6. $\frac{9}{11} \div 4$.
7. $\frac{20}{29} \div 5$.
8. $\frac{42}{43} \div 7$.
9. $\frac{50}{41} \div 25$.
10. $\frac{11}{16} \div 17$. [Art. 55.]
11. $\frac{5}{12} \div 12$.
12. $\frac{81}{15} \div 9$.

Written Exercises.

13. $\frac{15}{16} \div 6$.
14. $\frac{24}{25} \div 16$.
15. $\frac{36}{49} \div 30$.
16. $\frac{24}{25} \div 7$.
17. $2\frac{1}{5} \div 11$.
18. $7\frac{1}{5} \div 6$.
19. $9\frac{1}{7} \div 8$.
20. $\frac{25}{27} \div 15$.
21. $5\frac{3}{8} \div 5$.
22. $7\frac{4}{9} \div 6$.
23. $8\frac{3}{14} \div 15$.
24. $12\frac{2}{9} \div 11$.

25. $85\frac{3}{7} \div 9$. 26. $214\frac{1}{12} \div 7$. 27. $174\frac{15}{16} \div 18$.

28. $7\frac{11}{16} \div 15$. 29. $254\frac{7}{15} \div 25$.

123. Division by a Fraction.

If the fraction $\frac{7}{8}$ be divided by 1 (unity), the quotient is $\frac{7}{8}$; but, if the unit be separated into thirds, and one of these thirds be used (instead of unity) as the divisor, the quotient is 3 times as large as before.

Thus, $\frac{7}{8} \div 1 = \frac{7}{8}$; but $\frac{7}{8} \div \frac{1}{3} = \frac{7}{8} \times 3 = \frac{21}{8}$.

Now, if the second divisor ($\frac{1}{3}$) be multiplied by 2, the quotient ($\frac{21}{8}$) must be divided by 2; thus,

$$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \times 3 \div 2$$
$$= \frac{21}{8} \div 2 = \frac{21}{16};$$

i.e.,
$$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \times \frac{3}{2} = \frac{21}{16}.$$

The same reasoning will apply to all cases.

Hence, **to divide by a fraction, we must multiply by the fraction inverted.**

NOTE. *Sometimes* a short method of dividing a fraction by a fraction is to divide the numerator and denominator of the dividend by the numerator and denominator of the divisor, respectively; thus,

$$\frac{8}{9} \div \frac{4}{3} = \frac{2}{3}.$$

Ex. 1. *Divide* $\frac{5}{6}$ *by* $\frac{15}{32}$.

$$\frac{5}{6} \div \frac{15}{32} = \frac{\overset{1}{\cancel{5}}}{\underset{3}{\cancel{6}}} \times \frac{\overset{16}{\cancel{32}}}{\underset{3}{\cancel{15}}} = \frac{16}{9} = 1\frac{7}{9}.$$

Ex. 2. *Divide* $2\frac{1}{7}$ *by* $1\frac{13}{14}$.

The mixed numbers must first be expressed as improper fractions.

Then
$$\frac{15}{7} \div \frac{27}{14} = \frac{\overset{5}{\cancel{15}}}{\underset{1}{\cancel{7}}} \times \frac{\overset{2}{\cancel{14}}}{\underset{9}{\cancel{27}}} = \frac{10}{9} = 1\frac{1}{9}.$$

EXAMPLES XXXVI.

Written Exercises.

Simplify:

1. $\frac{3}{4} \div \frac{2}{3}$.
2. $\frac{5}{6} \div \frac{2}{3}$.
3. $\frac{2}{9} \div \frac{3}{4}$.
4. $\frac{1}{5} \div \frac{3}{7}$.
5. $\frac{1}{4} \div \frac{3}{4}$.
6. $\frac{7}{12} \div \frac{11}{12}$.
7. $\frac{3}{8} \div \frac{5}{8}$.
8. $\frac{7}{16} \div \frac{9}{16}$.
9. $\frac{1}{2} \div \frac{1}{4}$.
10. $\frac{1}{4} \div \frac{1}{2}$.
11. $\frac{2}{3} \div \frac{1}{6}$.
12. $\frac{4}{6} \div \frac{2}{3}$.
13. $2\frac{3}{4} \div 2\frac{1}{4}$.
14. $5\frac{1}{10} \div 1\frac{7}{10}$.
15. $6\frac{2}{9} \div 1\frac{5}{9}$.
16. $4\frac{11}{16} \div 1\frac{9}{16}$.
17. $6\frac{3}{5} \div 11$.
18. $6\frac{3}{7} \div 9$.
19. $\frac{75}{22} \div \frac{25}{11}$.
20. $2\frac{11}{12} \div 7$.
21. $\frac{25}{81} \div \frac{5}{27}$.
22. $\frac{13}{64} \div \frac{39}{68}$.
23. $\frac{64}{125} \div \frac{48}{55}$.
24. $\frac{121}{144} \div \frac{143}{180}$.
25. $2\frac{6}{25} \div 1\frac{1}{35}$.
26. $4\frac{4}{15} \div 1\frac{29}{39}$.
27. $1\frac{61}{64} \div 1\frac{7}{48}$.
28. $1\frac{23}{112} \div 1\frac{37}{143}$.
29. $\frac{143}{145} \div 2\frac{59}{68}$.
30. $5\frac{7}{8} \div 2\frac{13}{32}$.
31. $11\frac{22}{117} \div 12\frac{89}{351}$.
32. $2\frac{138}{187} \div 5\frac{111}{221}$.

124. When unity is divided by *any* number, the quotient is called the **Reciprocal** of the number; thus,

$\frac{1}{5}$ is the reciprocal of 5; $\frac{4}{3}$ is the reciprocal of $\frac{3}{4}$; 5 is the reciprocal of $\frac{1}{5}$.

Any number $\times$ *its reciprocal* $= 1$.

125. A fraction of a fraction is called a **Compound Fraction.**

Thus, $\frac{2}{3}$ of $\frac{5}{7}$ is a compound fraction.

To take $\frac{2}{3}$ of $\frac{5}{7}$, we must divide $\frac{5}{7}$ into 3 equal parts and take 2 of those parts.

Hence, $\frac{2}{3}$ of $\frac{5}{7}$ is the same as $\frac{5}{7} \times \frac{2}{3}$.

Ex. 1. *Multiply* $\frac{3}{7}$ *of* $2\frac{1}{5}$ *by* $\frac{5}{9}$ *of* $1\frac{3}{4}$.

$$\tfrac{3}{7} \text{ of } 2\tfrac{1}{5} = \tfrac{3}{7} \times \tfrac{11}{5}, \text{ and } \tfrac{5}{9} \text{ of } 1\tfrac{3}{4} = \tfrac{5}{9} \times \tfrac{7}{4};$$

hence the required product $= \frac{\cancel{3}}{\cancel{7}} \times \frac{11}{\cancel{5}} \times \frac{\cancel{5}}{\underset{3}{\cancel{9}}} \times \frac{\cancel{7}}{4} = \frac{11}{12}$.

EXAMPLES XXXVII.

Written Exercises.

1. State the reciprocals of 12, $\frac{4}{9}$, $\frac{18}{11}$, $\frac{1}{6}$, and $\frac{41}{42}$.

Simplify:

2. $\frac{4}{5}$ of $\frac{5}{6}$.
3. $\frac{7}{8}$ of $\frac{8}{9}$.
4. $\frac{5}{6}$ of $\frac{6}{25}$.
5. $1\frac{3}{4}$ of $2\frac{2}{7}$.
6. $3\frac{1}{5}$ of $6\frac{1}{4}$.
7. $7\frac{7}{8}$ of $2\frac{2}{9}$.
8. $3\frac{1}{5}$ of $3\frac{1}{8}$.
9. $\frac{5}{6}$ of $\frac{2}{7}$ of $\frac{7}{10}$.
10. $2\frac{1}{2}$ of $3\frac{1}{4}$ of $\frac{4}{5}$.
11. $6\frac{2}{7}$ of $2\frac{3}{11}$ of $1\frac{2}{5}$.
12. $\frac{2}{3}$ of $\frac{5}{9} \times \frac{9}{25}$ of $2\frac{1}{4}$.
13. $\frac{7}{9}$ of $2\frac{3}{5} \times 1\frac{3}{13}$ of $2\frac{1}{8}$.
14. $1\frac{1}{7}$ of $3\frac{1}{5} \times \frac{15}{16} \times 5\frac{1}{4}$.
15. $1\frac{4}{5}$ of $3\frac{3}{4} \times 5\frac{1}{3}$ of $7\frac{1}{9}$.

126. A fraction whose numerator, or denominator, or both, are fractional is called a **Complex Fraction**.

Thus, $\dfrac{\frac{3}{5}}{7}$, $\dfrac{\frac{2}{3}}{\frac{5}{7}}$, and $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{3}{4}+\frac{1}{2}}$ are complex fractions.

Complex fractions are simplified by dividing the numerator (simplified) by the denominator (simplified).

Ex. 1. *Simplify* $\dfrac{\frac{2}{3}}{\frac{5}{7}}$.

$$\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15}.$$

Ex. 2. *Simplify* $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{3}{4}+\frac{1}{2}}$.

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{3}{4}+\frac{1}{2}} = (\tfrac{1}{2}+\tfrac{1}{3}) \div (\tfrac{3}{4}+\tfrac{1}{2})$$
$$= \tfrac{5}{6} \div \tfrac{5}{4}$$
$$= \frac{\cancel{5}}{\underset{3}{\cancel{6}}} \times \frac{\overset{2}{\cancel{4}}}{\cancel{5}} = \frac{2}{3}.$$

CAUTION. Dividing by the sum of two fractions is not equivalent to multiplying by the sum of the reciprocals of those fractions.

In solving the above example the following would be wrong :

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{3}{4}+\frac{1}{2}}=(\tfrac{1}{2}+\tfrac{1}{3})\times(\tfrac{4}{3}+\tfrac{2}{1}).$$

A complex fraction is unchanged in value by multiplying its numerator and denominator by the same number.

For example, $\dfrac{\frac{5}{7}}{\frac{4}{9}}=\dfrac{\frac{5}{7}\times 11}{\frac{4}{9}\times 11}.$

For $\qquad \frac{5}{7}\div\frac{4}{9}=\frac{5}{7}\times\frac{9}{4}=\frac{45}{28};$

and $\qquad \dfrac{5}{7}\times 11\div\dfrac{4}{9}\times 11=\dfrac{\overset{5}{\cancel{55}}}{7}\times\dfrac{9}{\underset{4}{\cancel{44}}}=\dfrac{45}{28}.$

Ex. 1. *Simplify* $\dfrac{\frac{3}{4}-\frac{2}{3}}{\frac{7}{8}-\frac{5}{6}}.$

Multiply the numerator and denominator by 24, the L.C.M. of 3, 4, 6, 8. Then we have

$$\frac{(\frac{3}{4}-\frac{2}{3})\,24}{(\frac{7}{8}-\frac{5}{6})\,24}=\frac{18-16}{21-20}=\frac{2}{1}=2.$$

Ex. 2. *Simplify* $\dfrac{3}{5+\dfrac{2}{7-\dfrac{3}{4+\frac{1}{2}}}}.$

$$\frac{3}{5+\dfrac{2}{7-\dfrac{3}{4+\frac{1}{2}}}}=\frac{3}{5+\dfrac{2}{7-\frac{6}{9}}}=\frac{3}{5+\frac{18}{57}}=\frac{171}{303}=\frac{57}{101}.$$

First, multiply the numerator and denominator of the lowest complex fraction, namely $\dfrac{3}{4+\frac{1}{2}}$, by 2, and we get $\frac{6}{9}$. Next, multiply the numerator and denominator of the fraction $\dfrac{2}{7-\frac{6}{9}}$ by 9, and we get $\frac{18}{57}$. Then multiply the numerator and denominator of $\dfrac{3}{5+\frac{18}{57}}$ by 57, and we get $\frac{171}{303}$, which is then reduced to its lowest terms.

A fraction of this type is called a **Continued Fraction.**

EXAMPLES XXXVIII.

Written Exercises.

Simplify:

1. $\dfrac{5}{\frac{5}{6}}$.

2. $\dfrac{7}{\frac{3}{4}}$.

3. $\dfrac{\frac{3}{4}}{6}$.

4. $\dfrac{1\frac{2}{3}}{10}$.

5. $\dfrac{\frac{2}{3}}{\frac{4}{9}}$.

6. $\dfrac{\frac{4}{9}}{\frac{2}{3}}$.

7. $\dfrac{\frac{11}{12}}{\frac{5}{6}}$.

8. $\dfrac{\frac{3}{7}}{\frac{5}{14}}$.

9. $\dfrac{2\frac{1}{5}}{4\frac{2}{5}}$.

10. $\dfrac{5\frac{1}{4}}{4\frac{1}{5}}$.

11. $\dfrac{6\frac{2}{9}}{1\frac{5}{9}}$.

12. $\dfrac{7\frac{3}{7}}{11\frac{5}{9}}$.

13. $\dfrac{\frac{2}{3}+\frac{1}{6}}{\frac{3}{4}+\frac{1}{8}}$.

14. $\dfrac{\frac{11}{12}-\frac{3}{8}}{\frac{11}{12}+\frac{3}{8}}$.

15. $\dfrac{\frac{8}{9}+\frac{3}{11}}{\frac{2}{9}+\frac{7}{11}}$.

16. $\dfrac{\frac{7}{12}-\frac{5}{13}}{\frac{8}{13}-\frac{5}{12}}$.

17. $\dfrac{3\frac{5}{8}-1\frac{7}{8}}{2\frac{3}{4}-\frac{5}{8}}$.

18. $\dfrac{7\frac{3}{11}-4\frac{6}{11}}{5\frac{7}{12}-2\frac{8}{9}}$.

19. $\dfrac{15\frac{3}{14}-10\frac{10}{21}}{18\frac{5}{7}-15\frac{9}{56}}$.

20. $\dfrac{2\frac{3}{4}-1\frac{5}{8}}{\frac{3}{4}\text{ of }\frac{8}{9}}$.

21. $\dfrac{3\frac{1}{2}\text{ of }4\frac{1}{3}}{2\frac{1}{3}\text{ of }6\frac{1}{2}}$.

22. $\dfrac{\frac{3}{5}\div 2\frac{1}{2}}{1\frac{1}{15}\div 2\frac{2}{5}}$.

23. $\dfrac{3\frac{1}{4}+4\frac{1}{3}}{6\frac{1}{2}+1\frac{1}{12}}$.

24. $\dfrac{\frac{1}{2}+\frac{1}{4}+\frac{1}{6}}{\frac{8}{25}+\frac{9}{100}+\frac{7}{50}}$.

25. $\dfrac{\frac{14}{15}+\frac{13}{14}-\frac{9}{10}-\frac{8}{9}}{\frac{1}{10}+\frac{1}{9}-\frac{1}{15}-\frac{1}{14}}$.

26. $\dfrac{22}{1+\dfrac{5}{8+\frac{3}{4}}}$.

27. $\dfrac{35}{5-\dfrac{4}{7-\frac{3}{5}}}$.

28. $\dfrac{5\frac{1}{2}-3\frac{3}{4}}{\dfrac{2}{5}+\dfrac{19}{3\frac{1}{2}+2\frac{1}{5}}}$.

29. $\dfrac{1}{2-\dfrac{1}{3+\dfrac{1}{2-\frac{1}{3}}}}$.

30. $\dfrac{2}{1+\dfrac{3}{2+\dfrac{4}{3+\frac{5}{4}}}}$.

31. $\dfrac{7}{5+\dfrac{3}{1-\dfrac{1}{3-\frac{5}{7}}}}$.

32. $\dfrac{7\frac{1}{2}}{3+\dfrac{2}{5-\dfrac{2}{1-\frac{1}{7}}}}$.

33. $\dfrac{3\frac{4}{11}}{3\frac{1}{2}-\dfrac{2}{3-\dfrac{3}{4+\frac{1}{2}}}}$.

127. We now proceed to give examples of a more complicated nature; it will be well, however, for the student to consider carefully the following cases in which mistakes are frequently made in the meaning of the signs employed.

I. Operations of multiplication and division are to be performed in order from left to right, and each sign is a direction to multiply or divide what precedes the sign by **the number** that follows **next after it.**

For example, $36 \times 6 \div 3 = 216 \div 3 = 72$,

$$36 \div 6 \div 3 = 6 \div 3 = 2,$$

and $36 \div 6 \times 3 = 6 \times 3 = 18.$

So also, $\frac{2}{3} \times \frac{3}{5} \div \frac{8}{15} = \frac{2 \times 3}{3 \times 5} \div \frac{8}{15} = \frac{2 \times 3}{3 \times 5} \times \frac{15}{8} = \frac{3}{4}$,

$$\frac{2}{3} \div \frac{5}{6} \div \frac{4}{5} = \frac{2}{3} \times \frac{6}{5} \div \frac{4}{5} = \frac{2}{3} \times \frac{6}{5} \times \frac{5}{4} = 1,$$

and $\frac{2}{3} \div \frac{5}{6} \times \frac{4}{5} = \frac{2}{3} \times \frac{6}{5} \times \frac{4}{5} = \frac{16}{25}$.

II. Numbers connected by the sign '**of**' must be considered as a **single number**, just as if they were enclosed in brackets.

Thus, $\frac{14}{15} \div \frac{2}{5}$ of $\frac{7}{8} = \frac{14}{15} \div \frac{2 \times 7}{5 \times 8} = \frac{14}{15} \times \frac{5 \times 8}{2 \times 7} = \frac{8}{3}$.

Again, $\frac{3}{4}$ of $\frac{5}{8} \div \frac{3}{8}$ of $\frac{5}{6} = \frac{3 \times 5}{4 \times 8} \div \frac{3 \times 5}{8 \times 6} = \frac{3 \times 5}{4 \times 8} \times \frac{8 \times 6}{3 \times 5} = \frac{3}{2}$.

III. Before performing any operations of addition or subtraction, all multiplications and divisions must be performed, and complex and compound fractions must be reduced to simple fractions.

Thus, $\frac{2}{3}+\frac{3}{4}$ of $\frac{5}{6}+\frac{3}{4}=\frac{2}{3}+\frac{3\times 5}{4\times 6}+\frac{3}{4}$

$$=\frac{2}{3}+\quad\frac{5}{8}\quad+\frac{3}{4}=\frac{16+15+18}{24}=\frac{49}{24}=2\tfrac{1}{24}.$$

It is a very common mistake to work a question of this kind as if it meant

$$(\tfrac{2}{3}+\tfrac{3}{4}) \text{ of } (\tfrac{5}{6}+\tfrac{3}{4}).$$

EXAMPLES XXXIX.

Written Exercises.

Simplify:

1. $\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}$.
2. $\frac{2}{3}\div\frac{3}{4}\times\frac{4}{5}$.
3. $\frac{2}{3}\div\frac{3}{4}\div\frac{4}{5}$.
4. $\frac{5}{8}\div\frac{7}{12}\div\frac{5}{7}$.
5. $1\frac{1}{8}\div\frac{1}{2}\div2\frac{1}{4}$.
6. $1\frac{1}{8}\div\frac{1}{2}\times2\frac{1}{4}$.
7. $\frac{2}{3}\div\frac{5}{6}\times\frac{6}{35}\div\frac{2}{7}$.
8. $\frac{5}{12}\div\frac{3}{7}\times\frac{9}{14}\div3\frac{1}{8}$.
9. $6\frac{2}{3}\times4\frac{3}{8}\div5\frac{1}{4}\div2\frac{2}{3}$.
10. $\frac{2}{3}\div\frac{5}{6}$ of $\frac{3}{10}$.
11. $\frac{5}{7}$ of $\frac{3}{4}\div\frac{9}{14}$.
12. $\frac{3}{7}$ of $3\frac{1}{3}\div\frac{5}{7}$.
13. $\frac{5}{12}$ of $3\frac{3}{7}\div2\frac{6}{7}$.
14. $2\frac{1}{5}\div1\frac{3}{8}$ of $3\frac{1}{5}$.
15. $284\frac{3}{4}$ of $\frac{16}{19}\div17\frac{12}{19}$.
16. $\frac{4}{15}\div1\frac{1}{7}$ of $4\frac{1}{5}$ of $1\frac{7}{9}$.
17. $\frac{2}{3}$ of $\frac{9}{8}\div13\frac{1}{2}$ of $\frac{4}{9}$.
18. $4\frac{3}{4}$ of $1\frac{3}{8}\div4\frac{1}{8}$ of $2\frac{1}{9}$.
19. $2\frac{1}{5}\div\frac{5}{6}$ of $\frac{24}{25}\div5\frac{1}{2}$.
20. $1\frac{1}{4}\div3\frac{1}{4}$ of $\frac{3}{5}\times6\frac{1}{2}$.
21. $2\frac{1}{3}\div1\frac{5}{7}$ of $1\frac{1}{15}\div1\frac{3}{32}$.
22. $\frac{1}{2}+\frac{1}{3}$ of $\frac{3}{4}-\frac{3}{8}$.
23. $\frac{1}{2}$ of $\frac{1}{3}+\frac{3}{4}-\frac{3}{8}$.
24. $\frac{1}{3}-\frac{1}{4}$ of $\frac{1}{5}-\frac{1}{6}$.
25. $\frac{1}{3}$ of $\frac{1}{4}-\frac{1}{5}$ of $\frac{1}{6}$.
26. $2\frac{1}{2}+1\frac{1}{4}$ of $2\frac{1}{5}-3\frac{7}{8}$.
27. $2\frac{1}{2}$ of $1\frac{1}{4}+2\frac{1}{5}$ of $3\frac{7}{8}$.
28. $3\frac{4}{5}-\frac{3}{10}$ of $2\frac{1}{4}-1\frac{1}{8}$.
29. $\frac{5}{9}$ of $3\frac{3}{5}-2\frac{1}{5}$ of $\frac{5}{16}$ of $2\frac{3}{11}$.
30. $1\frac{1}{2}$ of $\frac{7}{9}\div3\frac{1}{9}-\frac{5}{16}$.
31. $\frac{5}{4}$ of $2\frac{2}{3}-4\frac{1}{3}$ of $5\frac{5}{6}\div2\frac{1}{3}$ of $3\frac{1}{4}$.

32. $\frac{5}{8}+1\frac{1}{6}$ of $2\frac{1}{7}-\frac{1}{2}\div\frac{4}{25}$.

33. $3\frac{1}{2}$ of $1\frac{1}{14}+7\frac{1}{2}-1\frac{1}{4}\div\frac{2}{3}$ of $\frac{5}{21}$.

34. $\left(\dfrac{5}{7}+\dfrac{11}{2\frac{1}{5}}\text{ of } 2\tfrac{3}{5}-\dfrac{27}{35}\right)\div 3\frac{171}{245}$.

35. $\dfrac{2\frac{1}{4}-\frac{2}{3}\text{ of }1\frac{5}{6}}{\frac{1}{5}\text{ of }3\frac{1}{3}+\frac{13}{36}}$.

36. $\dfrac{\frac{1}{2}-\frac{3}{10}}{\frac{2}{3}+1\frac{13}{21}}\div\dfrac{\frac{3}{2}\text{ of }\frac{4}{15}}{2\div\frac{7}{8}}$.

37. $\dfrac{\frac{1}{3}+\frac{1}{7}+\frac{5}{63}}{\frac{1}{2}-\frac{1}{18}}\div\dfrac{\frac{5}{14}\text{ of }2\frac{1}{4}}{\frac{2}{7}\text{ of }4\frac{1}{2}}$.

38. $\dfrac{\frac{1}{27}+\frac{1}{5}\text{ of }\frac{2}{5}\text{ of }\frac{4}{5}}{\frac{4}{25}-\frac{1}{3}\text{ of }\frac{2}{5}+\frac{1}{9}}\div\left(\frac{7}{9}-\frac{8}{15}\right)$.

39. $\dfrac{2\frac{1}{2}-\frac{1}{3}\text{ of }1\frac{1}{5}+\frac{4}{5}}{\frac{3}{5}+\frac{1}{3}\text{ of }\frac{5}{8}\div\frac{4}{5}\div 2\frac{1}{12}}$.

128. **To express one number or quantity as a fraction of another,** we proceed as follows:

Ex. 1. *Express* 174 *as a fraction of* 188.

Now
$$1=\tfrac{1}{188}\text{ of }188;$$
$$\therefore\ 174=\tfrac{174}{188}\text{ of }188$$
$$=\tfrac{87}{94}\text{ of }188.$$

Ex. 2. *Express* $2\frac{1}{2}$ *dollars as a fraction of* 8 *dollars.*

Now
$$1\text{ dollar}=\tfrac{1}{8}\text{ of 8 dollars};$$
$$\therefore\ 2\tfrac{1}{2}\text{ dollars}=\frac{2\frac{1}{2}}{8}\text{ of 8 dollars}$$
$$=\tfrac{5}{16}\text{ of 8 dollars.}$$

That number or quantity which is the part must be the *numerator*, while the other number must be the *denominator*, of the required fraction.

EXAMPLES XL.

Oral Exercises.

1. Express 27 as a fraction of 81.

2. Express 140 pounds as a fraction of 280 pounds.

What fraction of

3. 9 is 3?	6. 49 is 7?	9. 9 is $2\frac{1}{2}$?
4. 11 is 7?	7. 56 is 49?	10. 16 is $2\frac{1}{5}$?
5. 20 is 5?	8. 88 is 4?	11. $2\frac{1}{4}$ is $\frac{8}{11}$?

Written Exercises.

12. How many times does $8\frac{1}{2}$ feet contain $2\frac{1}{3}$ feet?

13. Express $\frac{3}{20}$ of 4 dollars as a part of 7 dollars.

14. Reduce $2\frac{1}{8}$ of 11 cents to the fraction of $5\frac{4}{9}$ of 15 cents.

15. What would be the measure of $\frac{2}{9}$ of 23 tons, if $\frac{1}{3}$ of 4 tons were used as the unit?

16. If the income of A is $\frac{2}{7}$ of $\frac{4}{9}$ of 1260 dollars, and the income of B is $\frac{8}{17}$ of $\frac{1}{32}$ of 5440 dollars, how large is A's income compared with B's? How large is B's income compared with A's?

What fraction of

17. $(8-\overline{2+3})(6+7-3^2)$ is 2^3?

18. $\dfrac{16\times 17}{\frac{1}{2}(20\frac{2}{3}-6\frac{2}{3}-2)}$ is $4[6-\{11-(3+1\frac{1}{2})\}+2]$?

19. $\dfrac{488\times 11-\frac{2}{3}\text{ of }7500}{3^2\times 5+1}$ is $(6+14)\div\frac{2}{3}$ of $\frac{15}{2}\times\frac{1}{9}\,\overline{22-5}$.

129. Reduction of Decimals to Common Fractions. Decimals may be considered as fractions with powers of 10 for denominators.

Thus, $.5=\frac{5}{10}$; $.56=\frac{56}{100}$; $.002007=\frac{2007}{1000000}$.

Ex. 1. *Reduce* .76 *to a common fraction.*

$$.76 = \frac{76}{100} = \frac{19}{25}.$$

Ex. 2. *Reduce* 4.012 *to a mixed number.*

$$4.012 = 4 + \frac{12}{1000} = 4\frac{3}{250}.$$

130. Reduction of Common Fractions to Decimals.

Ex. 1. *Express* $\frac{4}{25}$ *as a decimal.*

Since $\frac{4}{25}$ may be considered as the quotient obtained by dividing 4 by 25, we have only to perform this division. Thus,

```
25)4.00(.16
   2 5
   ----
   1 50
   1 50
   ----
```

Ex. 2. *Reduce, to 3 places of decimals, the common fractions,* $\frac{3}{4}$, $\frac{41}{48}$, *and* $\frac{6}{7}$; *and thus show that the fractions are in ascending order of magnitude.*

The decimals required are .75, .854..., and .857....

EXAMPLES XLI.

Oral Exercises.

Reduce to decimals:

1. $\frac{1}{2}$.	4. $\frac{3}{4}$.	7. $\frac{6}{8}$.	10. $\frac{7}{8}$.
2. $\frac{1}{4}$.	5. $\frac{3}{8}$.	8. $\frac{2.7}{9}$.	11. $\frac{3}{6}$.
3. $\frac{1}{8}$.	6. $\frac{2}{5}$.	9. $\frac{3}{5}$.	12. $\frac{11}{40}$.

Written Exercises.

13. $\frac{11}{25}$.	16. $\frac{11}{625}$.	19. $\frac{27}{512}$.	22. $7\frac{113}{1024}$.
14. $\frac{113}{125}$.	17. $\frac{16}{125}$.	20. $\frac{9}{16}$.	23. $13\frac{129}{312500}$.
15. $\frac{3}{125}$.	18. $\frac{21}{320}$.	21. $3\frac{19}{5120}$.	

CIRCULATING DECIMALS.

131. We have hitherto considered examples of division of decimals in which by proceeding far enough an exact quotient is found with no remainder. This, however, is by no means always the case; in fact, it is very rarely the case.

Consider, for example, the division of 5 by 3.

$$3 \underline{| 5.0000}$$
$$1.6666...$$

We may here continue the process of division to any extent, but each figure of the quotient will be 6, and the remainder will always be 2.

Again, divide 2 by 7.

$$7 \underline{| 2.000000000}$$
$$.285714285...$$

Here the six digits, 2, 8, 5, 7, 1, 4, come over and over again in the same order, and we shall never arrive at a stage at which there is no remainder.

When a decimal ends with digits which are repeated over and over again without end in the same order, the decimal is called a **Recurring** or **Circulating** decimal, and the digit, or set of digits, which is repeated, is called the **Circulating Period,** called also the **Repetend.**

Thus, 2.45555..., .014141414..., and 5.1246246246... are circulating decimals with circulating periods of one, two, and three figures, respectively.

A circulating period is denoted by placing dots over the first and last of the figures which recur.

Thus, $2.4\dot{5}$ denotes 2.45555..., $.0\dot{1}\dot{4}$ denotes .014141414..., and $5.1\dot{2}4\dot{6}$ denotes 5.1246246246...

A circulating decimal is said to be **Pure** or **Mixed,** according as all the figures *after the decimal point* do or do not recur.

Thus, $5.\dot{6}$, $31.\dot{2}\dot{4}$, and $14.\dot{1}3\dot{5}$ are *pure* circulating decimals; and $.5\dot{6}$, $3.1\dot{2}\dot{4}$, and $.14\dot{1}3\dot{5}$ are *mixed* circulating decimals.

A decimal which contains a definite number of figures is called a **Terminating** decimal, to distinguish it from a circulating decimal, which contains an unlimited number of figures.

NOTE. Although it is not possible to reduce *any* common fraction to a terminating decimal, it is always possible to find a decimal which is equal to the common fraction *to any degree of accuracy that may be required.*

For example, $\frac{1}{3}$ lies between .333 and .334, so that the difference between $\frac{1}{3}$ and .333 is less than one *one-thousandth*, so also the difference between $\frac{1}{3}$ and .333333 is less than one *one-millionth*; and so on.

Now there is no species of magnitude which can be measured with *perfect* accuracy. It would, for instance, be difficult to determine the length or the weight of a body without a possible error as great as one *one-thousandth* of the whole. Hence the measure of any quantity can be expressed as accurately by means of decimals as by means of fractions.

EXAMPLES XLII.

Written Exercises.

Express the following quotients as circulating decimals:

1. $1.5 \div 2.7$.
2. $10 \div .03$.
3. $1.7 \div .09$.
4. $.035 \div .072$.
5. $.316 \div 2.4$.
6. $.312 \div 8.8$.
7. $3.1 \div 7$.
8. $15.6 \div .07$.
9. $1.25 \div 13.2$.
10. $5.193 \div .0168$.
11. $.0235 \div .00616$.
12. $16.72 \div .0143$.
13. $.3157 \div .259$.
14. $27.31 \div 6.475$.
15. $693.11 \div .011396$.

Reduce the following common fractions to circulating decimals:

16. $\frac{1}{3}$.
17. $\frac{1}{7}$.
18. $\frac{5}{9}$.
19. $\frac{7}{11}$.
20. $\frac{6}{13}$.
21. $\frac{11}{14}$.
22. $\frac{8}{15}$.
23. $\frac{5}{17}$.

24. $2\frac{7}{22}$. 26. $5\frac{15}{28}$. 28. $11\frac{3}{56}$. 30. $2\frac{615}{616}$.

25. $3\frac{11}{21}$. 27. $7\frac{8}{31}$. 29. $13\frac{19}{168}$. 31. $5\frac{71}{518}$.

132. Reduction of a Circulating Decimal to an Equivalent Common Fraction.

We have seen (Art. 129) that a *terminating* decimal can be expressed as a common fraction. We have now to show that a *circulating* decimal may be expressed as a common fraction.

Consider the decimals, $.\dot{3}\dot{1}$, $.\dot{5}21\dot{6}$, and $.15\dot{6}0\dot{7}$.

In each case multiply the decimal by that power of 10 which will move the decimal point to the end of the first recurring period; also (if necessary) multiply the decimal by that power of 10 which will move the decimal point to the beginning of the first recurring period. Subtract the second product from the first, and notice the result.

$$\text{(i)}\qquad \begin{aligned} .\dot{3}\dot{1} \times 100 &= 31.\dot{3}\dot{1} \\ .\dot{3}\dot{1} \times \quad 1 &= \quad .\dot{3}\dot{1} \\ \hline .\dot{3}\dot{1} \times \ \ 99 &= 31. \\ \therefore\ .\dot{3}\dot{1} &= \frac{31}{99}. \end{aligned}$$

No advantage will be gained by repeating the $.\dot{3}\dot{1}$ in the minuend or subtrahend; we obtain only an integer in the remainder.

$$\text{(ii)}\qquad \begin{aligned} .\dot{5}21\dot{6} \times 10000 &= 5216.\dot{5}21\dot{6} \\ .\dot{5}21\dot{6} \times \qquad 1 &= \qquad .\dot{5}21\dot{6} \\ \hline .\dot{5}21\dot{6} \times \ \ 9999 &= 5216. \\ \therefore\ .\dot{5}21\dot{6} &= \tfrac{5216}{9999}. \end{aligned}$$

$$\text{(iii)}\qquad \begin{aligned} .15\dot{6}0\dot{7} \times 100000 &= 15607.\dot{6}0\dot{7} \\ .15\dot{6}0\dot{7} \times \qquad 100 &= \qquad 15.\dot{6}0\dot{7} \\ \hline .15\dot{6}0\dot{7} \times \ \ 99900 &= 15592. \\ \therefore\ .15\dot{6}0\dot{7} &= \tfrac{15592}{99900}. \end{aligned}$$

Three facts concerning the fraction equivalent to a circulating decimal are easily noted:

1. The numerator is the whole decimal minus the number expressed by the non-recurring digits.

2. The number of 9's in the denominator equals the number of recurring digits.

3. The number of naughts in the denominator equals the number of non-recurring digits.

Ex. 1. $.\dot{5} = \frac{5}{9}$.

Ex. 2. $.\dot{0}\dot{7} = \frac{7}{99}$.

Ex. 3. $.1\dot{5}\dot{6} = \frac{155}{990} = \frac{31}{198}$.

Ex. 4. $3.3\dot{1}\dot{2} = 3\frac{309}{990} = 3\frac{103}{330}$.

It should be noticed that by the above rule $.\dot{9} = \frac{9}{9} = 1$. This result can be seen independently; for the differences between 1 and the decimals, .9, .99, .999, etc., are respectively .1, .01, .001, etc., each difference being *one-tenth* of the preceding, and therefore when a large number of nines is taken, the difference between 1 and .99999... becomes inconceivably small.

Since $.\dot{9} = 1$, $.0\dot{9} = .1$, $.00\dot{9} = .01$, and so on, a recurring 9 can always be replaced by 1 in the next place to the left; for example, $.7\dot{9} = .8$ and $.24\dot{9} = .25$.

EXAMPLES XLIII.

Written Exercises.

Find common fractions in their lowest terms equivalent to the following circulating decimals:

1. $.\dot{3}$.
2. $.\dot{0}\dot{9}$.
3. $17.\dot{2}\dot{7}$.
4. $.\dot{1}\dot{5}$.
5. $1.\dot{0}2\dot{7}$.
6. $.\dot{0}3\dot{7}$.
7. $.\dot{1}8\dot{5}$.
8. $.\dot{3}9\dot{6}$.
9. $.\dot{1}4285\dot{7}$.
10. $.\dot{2}8571\dot{4}$.
11. $.\dot{4}2857\dot{1}$.
12. $.\dot{0}1298\dot{7}$.
13. $.\dot{0}487\dot{8}$.
14. $.\dot{0}731\dot{7}$.
15. $9.2\dot{3}$.
16. $.7\dot{9}$.
17. $6.3\dot{6}$.
18. $.3\dot{1}\dot{5}$.

19. $.11\dot{6}$.

20. $.02\dot{5}\dot{4}$.

21. $.01\dot{6}$.

22. $.74\dot{9}$.

23. $.20\dot{2}\dot{7}$.

24. $.19\dot{3}2\dot{4}$.

25. $.4\dot{0}243\dot{9}$.

26. $.3\dot{0}487\dot{8}$.

27. $11.30\dot{2}197\dot{6}$.

28. $.5\dot{4}2857\dot{1}$.

29. $.\dot{0}1234567\dot{9}$.

30. $.\dot{1}3580246\dot{9}$.

It should be noticed that *if a common fraction in its lowest terms be equivalent to a terminating decimal, the denominator of the fraction can contain only the prime factors 2 and 5.*

133. **Addition, Subtraction, Multiplication, and Division** of circulating decimals are performed after first reducing to common fractions. The answer in each case should be reduced to a circulating decimal.

134. An exact divisor of a number is sometimes called an **Aliquot Part** of the number.

$2\frac{1}{2}$ is an aliquot part of 10; $16\frac{2}{3}$ is an aliquot part of 100.

This enables us to use a short process of multiplication (or division) in cases where the multiplier (or divisor) is an *aliquot* part of some power of 10.

To $\times\ 3\frac{1}{3}$, $\times\ 10$ and $\div\ 3$.	To $\div\ 3\frac{1}{3}$, $\div\ 10$ and $\times\ 3$.
To $\times\ 12\frac{1}{2}$, $\times\ 100$ and $\div\ 8$.	To $\div\ 12\frac{1}{2}$, $\div\ 100$ and $\times\ 8$.
To $\times\ 16\frac{2}{3}$, $\times\ 100$ and $\div\ 6$.	To $\div\ 16\frac{2}{3}$, $\div\ 100$ and $\times\ 6$.
To $\times\ 25$, $\times\ 100$ and $\div\ 4$.	To $\div\ 25$, $\div\ 100$ and $\times\ 4$.
To $\times\ 33\frac{1}{3}$, $\times\ 100$ and $\div\ 3$.	To $\div\ 33\frac{1}{3}$, $\div\ 100$ and $\times\ 3$.
To $\times\ 125$, $\times\ 1000$ and $\div\ 8$.	To $\div\ 125$, $\div\ 1000$ and $\times\ 8$.

Read the signs 'multiply by' and 'divide by'.

135. **Square Roots of Fractions.**

Since
$$\left(\frac{3}{4}\right)^2=\frac{3\times 3}{4\times 4}=\frac{3^2}{4^2},$$
it follows conversely that
$$\sqrt{\frac{9}{16}}=\frac{3}{4}=\frac{\sqrt{9}}{\sqrt{16}}.$$

Thus, *the square root of a common fraction is equal to a fraction whose numerator and denominator are respectively the square roots of the numerator and denominator of the given fraction.*

Ex. 1. *Find the square roots of* $\frac{144}{169}$, $1\frac{9}{16}$, $.\dot{4}$, and $2.\dot{0}8641975\dot{3}$.

$$\sqrt{\frac{144}{169}}=\frac{\sqrt{144}}{\sqrt{169}}=\frac{12}{13};\quad \sqrt{1\tfrac{9}{16}}=\sqrt{\frac{25}{16}}=\frac{\sqrt{25}}{\sqrt{16}}=\frac{5}{4};$$

$$\sqrt{.\dot{4}}=\sqrt{\frac{4}{9}}=\frac{\sqrt{4}}{\sqrt{9}}=\frac{2}{3};$$

and

$$\sqrt{2.\dot{0}8641975\dot{3}}=\sqrt{2\tfrac{086419753}{999999999}}=\sqrt{2\tfrac{7}{81}}$$

$$=\sqrt{\frac{169}{81}}=\frac{\sqrt{169}}{\sqrt{81}}=\frac{13}{9}=1.\dot{4}.$$

Ex. 2. *Find, to four places of decimals,*

(i) $\sqrt{\frac{5}{9}}$, (ii) $\sqrt{\frac{4}{5}}$, (iii) $\sqrt{.\dot{3}}$, and (iv) $\frac{5}{\sqrt{3}}$.

(i) $\sqrt{\frac{5}{9}}=\frac{\sqrt{5}}{\sqrt{9}}=\frac{1}{3}\sqrt{5}$, which can be found as in Art. 88.

(ii) In examples in which the denominator is not a perfect square, the fraction should be expressed as a decimal. In the present case $\sqrt{\frac{4}{5}}=\sqrt{.8}=\ldots$, etc.

Or thus: $\sqrt{\frac{4}{5}}=\sqrt{\frac{20}{25}}=\frac{\sqrt{20}}{\sqrt{25}}=\frac{1}{5}\sqrt{20}=\ldots$, etc.

(iii) $.\dot{3}=.33'33'33'33'\ldots$ Then proceed as in Art. 88.

(iv) $\frac{5}{\sqrt{3}}=\frac{5\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}=\frac{5}{3}\sqrt{3}=\ldots$, etc.

The change of form from $\frac{5}{\sqrt{3}}$ to $\frac{5}{3}\sqrt{3}$ will save labor.

EXAMPLES XLIV.

Written Exercises.

Find the square roots of

1. $\frac{25}{144}$.
2. $1\frac{25}{144}$.
3. $\frac{169}{289}$.
4. $11\frac{14}{25}$.
5. $39\frac{1}{16}$.
6. $.\dot{1}$.
7. $.00\dot{4}$.
8. $.13\dot{4}$.
9. $1.36\dot{1}$.
10. $4.3820\dot{4}$.

Find, to four places of decimals, the square roots of

11. $\frac{7}{16}$.	13. $3\frac{1}{4}$.	15. $2.\dot{4}$.	17. $.0\dot{8}\dot{3}$.
12. $\frac{4}{11}$.	14. $8\frac{2}{7}$.	16. $.0\dot{4}\dot{1}$.	18. $3.5\dot{1}6\dot{2}$.

136. The H.C.F. and L.C.M. of Fractions.

By the H.C.F. of two or more fractions we mean a fractional H.C.F. The *quotients*, however, are *integral.*

A fraction ÷ a fraction = an *integer* only when the numerator and denominator of the dividend divided by the numerator and denominator of the divisor respectively, produce an integer and the reciprocal of an integer; thus,

$$\frac{14}{27} \div \frac{2}{81} = \frac{7}{\frac{1}{3}} = 21.$$

Here, $14 \div 2$ is an integer, and $27 \div 81$ is the reciprocal of an integer; *i.e.*, the numerator of the divisor is a factor and the denominator of the divisor is a multiple; also, the numerator of the dividend is a multiple, and the denominator of the dividend is a factor.

Hence, *the H.C.F. of two or more fractions must have for its numerator the H.C.F. of the given numerators, and for its denominator the L.C.M. of the given denominators.*

Also, *the L.C.M. of two or more fractions must have for its numerator the L.C.M. of the given numerators, and for its denominator the H.C.F. of the given denominators.*

NOTE. Before obtaining the H.C.F. or the L.C.M., the given fractions must be in their lowest terms, and mixed numbers must be reduced to improper fractions. The *L.C.M.* may be integral.

Ex. 1. $\begin{cases} \text{The H.C.F. of } \frac{8}{9} \text{ and } \frac{14}{15} = \frac{2}{45}; \\ \text{The L.C.M. of } \frac{8}{9} \text{ and } \frac{14}{15} = \frac{56}{3}. \end{cases}$

Ex. 2. $\begin{cases} \text{The H.C.F. of } \frac{5}{6} \text{ and } \frac{3}{5} = \frac{1}{30}; \\ \text{The L.C.M. of } \frac{5}{6} \text{ and } \frac{3}{15} = 5. \end{cases}$

EXAMPLES XLV.

Written Exercises.

Find H.C.F. and L.C.M. of

1. $\frac{4}{9}$, $\frac{8}{27}$, and $\frac{16}{3}$.
2. $\frac{7}{13}$, $\frac{6}{65}$, and $1\frac{4}{5}$.
3. $\frac{1}{2}$, $\frac{5}{6}$, $\frac{7}{8}$, and $\frac{11}{12}$.
4. $\frac{16}{21}$, $\frac{24}{35}$, and $\frac{32}{49}$.
5. $\frac{221}{374}$ and $\frac{247}{357}$.
6. $\frac{45}{76}$ and $1\frac{47}{58}$.
7. $\frac{48}{385}$, $\frac{60}{133}$, and $39\frac{3}{35}$.

EXAMPLES XLVI.

Miscellaneous Examples. Chap. IV.

1. Reduce $5\frac{1}{7}$, $8\frac{3}{11}$, and $25\frac{3}{25}$ to improper fractions.

2. Simplify $\frac{3}{4} - \frac{3}{8} + \frac{5}{12} - \frac{5}{16} + \frac{7}{24} - \frac{7}{48}$.

3. What must be added to $5\frac{7}{8}$ that the sum may be $12\frac{5}{6}$?

4. Multiply $2\frac{2}{3}$ of $5\frac{3}{8}$ by $3\frac{1}{7} \div 6\frac{1}{7}$.

5. Simplify $\dfrac{3\frac{1}{4} - \frac{5}{6} \times \frac{6}{7} + \frac{3}{14}}{\frac{3}{14} + \frac{1}{3} \text{ of } \frac{5}{4} - \frac{5}{21}}$.

---

6. Arrange, in ascending order of magnitude, the fractions, $\frac{7}{12}$, $\frac{5}{9}$, $\frac{8}{15}$, $\frac{13}{20}$.

7. From the sum of $\frac{1}{2}$ and $\frac{1}{3}$ take the difference between $\frac{1}{4}$ and $\frac{1}{6}$.

8. Simplify $2\frac{2}{5}$ of $4\frac{3}{8}$ of $5\frac{1}{7}$.

9. Simplify $\dfrac{3\frac{1}{2} + 2\frac{1}{3} - 4\frac{5}{6}}{\frac{7}{12} \div \frac{2}{3}} \times \frac{5}{16}$.

10. What fraction of 350 equals $\frac{3}{4}$ of 168?

11. Reduce $\frac{41020}{75008}$ and $\frac{25194}{88179}$ to their lowest terms.

12. Reduce to a common denominator $\frac{1}{504}$, $\frac{2}{819}$, $\frac{1}{468}$, and $\frac{3}{728}$.

13. Simplify $3\frac{3}{4} + 2\frac{2}{5}$ of $1\frac{1}{9} - 4\frac{1}{5}$.

14. Simplify $\dfrac{\frac{1}{5} - \frac{1}{6} + \frac{1}{9} - \frac{1}{18}}{\frac{1}{4} - \frac{1}{5} + \frac{1}{9} - \frac{2}{45}}$.

15. A and B started on a tour with 192 and 156 dollars respectively, and they had equal sums left at the end. A spent $\frac{7}{8}$ of his money; what fraction did B spend of his?

---

16. Add $\frac{1}{12}$, $\frac{5}{64}$, $\frac{7}{80}$, $\frac{13}{160}$, $\frac{27}{320}$, and $\frac{41}{480}$.

17. Subtract $5\frac{13}{15}$ from $7\frac{9}{20}$; also, $5\frac{3}{8} + 2\frac{5}{6}$ from $12\frac{1}{4}$.

18. Divide $2\frac{2}{15} + 2\frac{27}{35} - 3\frac{4}{21}$ by $2\frac{5}{21} + 3\frac{1}{3} - 4\frac{6}{7}$.

19. Simplify $\dfrac{1\frac{1}{2} + 2\frac{1}{4} \text{ of } 5\frac{1}{3} - 12\frac{3}{4}}{\frac{3}{4} + \frac{4}{5} \times 8\frac{3}{5} - 6\frac{1}{2}}$.

20. What is the value of $\frac{2}{3}$ of a property, if $\frac{5}{6}$ of it is worth 750 dollars?

---

21. Reduce $\frac{3472}{6541}$, $\frac{10010}{20592}$, and $\frac{17952}{25245}$ to their lowest terms.

22. Show that $\dfrac{1}{2} + \dfrac{1}{2 \times 3 \times 4} + \dfrac{1}{2 \times 3 \times 4 \times 5 \times 6}$ is less than $\frac{6}{11}$, but greater than $\frac{8}{15}$.

23. Simplify $\frac{235}{231} \times \frac{66}{65} \times \frac{91}{94}$.

24. Simplify $\dfrac{2\frac{1}{4} + \frac{2}{3} \text{ of } 1\frac{1}{2} - 3\frac{1}{8}}{\frac{3}{4} + \frac{1}{4} \div \frac{2}{3} - \frac{1}{8}}$.

25. Find the G.C.M. of $5\frac{1}{3}$ and $4\frac{4}{5}$; express the answer as a circulating decimal and obtain the square root.

26. Simplify $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} - \frac{13}{6} - \frac{1}{20}$.

27. Subtract $2\frac{33}{34}$ from $5\frac{47}{51}$, and $8\frac{16}{45}$ from $12\frac{7}{60}$.

28. Multiply $3\frac{1}{8}$ of $5\frac{1}{7}$ by $4\frac{1}{5}$ of $\frac{3}{2}$, and divide the result by $4\frac{1}{2}$ of $1\frac{7}{8}$.

29. Simplify $\dfrac{\frac{2}{3}+\frac{3}{7}}{\frac{3}{4}+\frac{2}{5}} \div \dfrac{2\frac{2}{3}}{\frac{3}{4}-\frac{2}{5}}$, and $\dfrac{8\frac{2}{17} \text{ of } (\frac{7}{6}-\frac{6}{7})}{(8-\frac{1}{3}) \div 6\frac{4}{5} \text{ of } (1-\frac{4}{39})}$.

30. Find the L.C.M. of $\frac{1}{2}$ and $\frac{2}{3}$.

31. By how much does the sum of $1\frac{1}{6}$, $\frac{5}{8}$, and $\frac{1}{10}$ fall short of the sum of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{11}{12}$?

32. Simplify $2\frac{1}{2} - \frac{1}{3}$ of $4\frac{1}{5} + 2\frac{1}{5} \div 3\frac{1}{5} \times 2\frac{1}{4} - \frac{47}{320}$.

33. How many pieces each $\frac{3}{5}$ of 1 inch can be cut from a wire whose length is $5\frac{1}{2}$ inches; and what will be the length of the piece left over?

34. Simplify $\dfrac{4\frac{1}{2} - 3\frac{1}{3} + 5\frac{1}{12}}{7\frac{1}{2} - 4\frac{1}{3} + 11\frac{1}{12}} - \dfrac{11\frac{3}{4} - 5\frac{7}{15}}{11\frac{1}{2} + 5\frac{3}{5}}$.

35. Find L.C.M. and H.C.F. of $\frac{63}{209}$, $\frac{117}{110}$, and $\frac{90}{1463}$.

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36. Take the sum of $\frac{3}{4}$ and $\frac{1}{2}$ from the sum of $\frac{5}{6}$ and $\frac{2}{3}$.

37. After taking away $\frac{1}{2}$ and $\frac{2}{5}$ of a certain quantity, what fraction of the whole will be left?

38. Multiply $1\frac{1}{3} + 3\frac{4}{5}$ by $3\frac{1}{2} + 2\frac{4}{7}$, and divide the result by $5\frac{1}{2}$ of $5\frac{2}{3}$.

39. Simplify $\dfrac{2\frac{69}{88} - \frac{5}{13} \text{ of } 4\frac{8}{11} + 3\frac{11}{12} \div 7\frac{5}{6}}{3\frac{3}{11} \text{ of } 10\frac{1}{12} - 3\frac{1}{3} \div 4\frac{4}{9}}$.

40. Find the value of $3\frac{3}{11}$ of $4\frac{5}{7}$ of $1\frac{5}{9}$ of $\frac{1}{24}$.

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41. Add $\frac{2}{5}$, $\frac{35}{80}$, $\frac{14}{100}$, $\frac{3}{140}$, and $\frac{3}{2800}$.

42. Simplify $\dfrac{5}{\sqrt{3}}$.

43. Simplify $\dfrac{2\frac{1}{4} - \frac{2}{3} \text{ of } 1\frac{5}{6}}{\frac{1}{5} \text{ of } 3\frac{1}{3} + \frac{7}{8}}$, and $\dfrac{1}{1 - \dfrac{1}{2 + \dfrac{1}{3 - \frac{1}{4}}}}$.

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44. There are three partners in a certain business, one of whom provided $\frac{2}{5}$ of the whole capital, and another provided $\frac{3}{8}$. What fraction of the whole was supplied by the third partner?

45. A man gives $\frac{1}{4}$ of his money to his wife, $\frac{1}{4}$ of the remainder to his son, and $\frac{1}{4}$ of what then remains to his daughter; and has still left a sum of 1350 dollars. How much was there at first?

46. Simplify $\dfrac{9\frac{1}{9}+8\frac{1}{8}-7\frac{1}{7}}{9\frac{1}{9}-6\frac{9}{56}+7\frac{1}{7}}$, and $\dfrac{1-\dfrac{1}{2-\frac{1}{3}}}{1+\dfrac{1}{2+\frac{1}{3}}}$.

47. Divide $1\frac{1}{2}$ of $5\frac{3}{8}$ by $2\frac{3}{4}$ of $7\frac{9}{11}$.

48. Simplify $\dfrac{4\frac{1}{2}-3\frac{1}{3}+5\frac{1}{12}}{7\frac{1}{2}-4\frac{1}{3}+11\frac{1}{12}}-\dfrac{11\frac{7}{10}-5\frac{7}{15}}{11\frac{1}{2}+5\frac{3}{5}}$.

49. Find the value of $\frac{2}{5}$ of $\frac{3}{8}$ of 5 dollars $-\ \frac{1}{7}$ of $\frac{5}{6}$ of 2 dollars, and express the difference as a fraction of 11.25 dollars.

---

50. Reduce to its simplest form

$$\frac{(3\frac{4}{5}+5\frac{1}{9}-\frac{1}{45})\,(4\frac{1}{5}-3\frac{1}{4})}{1\frac{5}{11}+2\frac{1}{8}-(2\frac{9}{16}-\frac{1}{8}-\frac{1}{22})}.$$

51. After spending $\frac{2}{5}$ of his money, a man found that $\frac{3}{7}$ of the remainder was 63 cents; how many cents had he at first?

52. I purchased some square tiles for a room 483 inches long and 266 inches broad; the manufacturer sent me the largest tiles I could use; how long was each tile?

53. A man travelled $\frac{3}{4}$ of a certain distance by railway, $\frac{4}{21}$ of the whole distance by coach, and walked the rest of the way, which was 15 miles. What was the length of the whole journey?

54. By what must $7\frac{2}{3}$ of $3\frac{3}{4}$ be multiplied that the product may equal $4\frac{3}{5}$ of $2\frac{5}{8}$?

55. Simplify $\dfrac{6\frac{3}{4} - \frac{3}{7} \text{ of } 15\frac{3}{4} + 2\frac{2}{35} \div 1\frac{11}{25}}{\frac{3}{4} \text{ of } 7\frac{3}{7} - 5\frac{3}{5} \div 3\frac{4}{15}}$; factor the answer into two fractions so that one factor shall be a perfect square.

---

56. Find the H.C.F. and the L.C.M. of $\frac{16}{35}$ and $\frac{20}{21}$.

57. Subtract $10\frac{11}{15}$ from $23\frac{3}{5}$, and $15\frac{23}{48}$ from $20\frac{39}{84}$.

58. Simplify $(3\frac{1}{7} + 2\frac{1}{4} + 4\frac{3}{5}) \div (\frac{1}{2} + \frac{2}{5})$ of $(\frac{1}{2} - \frac{3}{7})$.

59. Simplify

(i) $\dfrac{\frac{3}{4} - \frac{2}{3} \times \frac{1}{2} - \frac{1}{6}}{(4\frac{1}{2} - 2\frac{1}{3}) \div (3\frac{1}{3} - 1\frac{1}{6})}$, (ii) $\dfrac{3\frac{2}{13}}{2\frac{1}{2} - \dfrac{3}{4 - \frac{3}{4}}}$.

60. A man gives away $\frac{3}{5}$ of his money and afterwards $\frac{7}{8}$ of the remainder. What fraction of the whole had he then left?

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61. Reduce to a common denominator, and arrange in order of magnitude the fractions, $\frac{5}{12}$, $\frac{7}{15}$, $\frac{9}{20}$, $\frac{13}{32}$, $\frac{19}{48}$.

62. Multiply the difference between $3\frac{1}{2}$ of $1\frac{1}{14} + 7\frac{1}{2}$ and $2\frac{1}{7} \div \frac{2}{3}$ of $\frac{5}{21}$ by the sum of $\dfrac{1}{5\frac{1}{7}}$ and $\dfrac{1}{7\frac{1}{5}}$.

63. Simplify $\frac{1}{7}$ of $\dfrac{6\frac{7}{9} - 2\frac{2}{9}}{\frac{1}{9} \text{ of } 18\frac{1}{2}} \div \dfrac{\frac{2}{7}(\frac{64}{3} + 19\frac{2}{3})}{\frac{8}{11} + \frac{12}{13} \text{ of } 6\frac{1}{2}}$.

64. After spending $\frac{2}{5}$ of his money, a boy found that $\frac{4}{7}$ of the remainder was $2\frac{2}{5}$ dollars. What had he at first?

65. Reduce to their lowest terms $\frac{217}{403}$, $\frac{1496}{1530}$, and $\frac{28907}{154241}$.

66. A gave $\frac{1}{3}$ of his marbles to B, $\frac{1}{6}$ to C, $\frac{1}{8}$ to D, $\frac{1}{16}$ to E, and then had 105 left. How many did each receive?

67. Find $2\frac{31}{37}$ of $\{2\frac{15}{28} - \frac{4}{7}$ of $(3\frac{1}{9}$ of $2\frac{4}{7} - 5\frac{2}{3})$ of $1\frac{1}{5} - \frac{7}{12}\}$.

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68. 64 feet of brass rods cost $12\frac{1}{2}$ cents a foot; what was the cost of the rods?

69. NASHVILLE, TENN., Jan. 1, 1895.

J. S. CUSHING & Co. *To* H. A. ARMSTRONG, *Dr.*

For	*2*	*lb.*	*Sugar*	*@*	*7*	*cents*	
"	*5*	"	*Tea*	"	*50*	"	
"	*11*	"	*Coffee*	"	*34*	"	
"	*17*	"	*Starch*	"	*14*	"	

Find the amount of the above bill; answer in dollars and cents, letting 100 cents equal one dollar.

What would be the answer in dollars, and the decimal of a dollar?

70. Find the H.C.F. and L.C.M. of 1.485 and 12.6.

71. Simplify $\dfrac{16\frac{2}{3} \times 4 \times 8\frac{1}{2}}{8\frac{1}{3} \times 32 \times 34}$.

72. Add, without changing positions: 67.04, 12, $5\frac{6}{17}$, $9\frac{4}{17}$, 4.17, $243\frac{5}{17}$, 14, $8\frac{2}{17}$.

73. A certain lake is $.3\dot{2}\dot{7}$ of a mile long; what is its length compared with the length of a second lake $2\frac{2}{5}$ miles long?

(Answer must be reduced to a circulating decimal.)

74. Three tanks contain 924, 1500, and 2520 gallons of water respectively; what is the largest number of gallons that can run from each of the tanks per minute and allow all to be emptied in a whole number of minutes, the rate of flow from each tank being the same? How many minutes are required to empty each tank?

CHAPTER V.

DECIMAL MEASURES.

137. Anything which can be increased or diminished is called a **Magnitude.**

Lengths, areas, weights, etc., are magnitudes.

To **measure** a magnitude is to compare it with some known magnitude of the *same kind,* which is taken as a unit, and to say how many times the unit must be repeated in order to make up the magnitude in question.

For example, to measure any given length of string, is to find how many times some known length, say a foot, must be repeated to make up the given length; and this number of times is called the **measure** of the length.

A measured magnitude is called a **Quantity.**

Thus, any quantity is expressed by a *number* and *a unit of the same kind as itself.*

138. Numbers are first used in connection with distinct objects, and are afterwards used in measuring continuous magnitudes of any kind. If the continuous magnitude cannot be measured by *one* unit, a series of units smaller and smaller in value may be used.

For example, to measure a string, some definite length, say a yard, is fixed on as a unit. Suppose the given string contains $6\frac{1}{2}$ yards. We may use a second unit, say a foot, to measure the

$\frac{1}{2}$ yard. If there are 3 feet in one yard, the $\frac{1}{2}$ yard will be $1\frac{1}{2}$ feet, and the string will measure 6 yards $1\frac{1}{2}$ feet. This $\frac{1}{2}$ foot may be expressed in a smaller unit still, say an inch; if there are 12 inches in a foot, the $\frac{1}{2}$ foot will be 6 inches, and the string will measure 6 yards 1 foot 6 inches.

139. Quantities expressed in terms of a single unit are called **Simple Quantities,** and quantities which are expressed in terms of more than one unit are called **Compound Quantities.**

To measure every different kind of quantity, some standard unit is employed, and also other units which are obtained by subdivisions and repetitions of the standard unit.

Units which require 10 of one to make one of the next higher are the simplest to use. Such units are called **Decimal Units.**

In numeration of quantities, units of different kinds are called units of *different denominations.*

TABLES OF DECIMAL UNITS.

140.* **Table of United States Money.**

Money is a measure of values.

10 mills (m.)	= 1 cent (ct.).
10 cts.	= 1 dime (d.).
10 d.	= 1 dollar ($).
10 $	= 1 eagle (e.).

The *eagle* is usually called *ten dollars,* and the *dime* is usually called *ten cents;* so that the only names generally used are dollars and cents.

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* It is advisable to study numeration and notation of decimal measures at the same time.

Thus, \$25.35 is read, '25 dollars 35 cents,' and not, '2 eagles 5 dollars 3 dimes 5 cents'; also, \$.20 is read, '20 cents.'

The notation is as follows:

The figure representing		eagles	is put in		tens' place,
"	"	dollars	"	"	units' place,
"	"	dimes	"	"	tenths' place,
"	"	cents	"	"	hundredths' place,
"	"	mills	"	"	thousandths' place.

141. A sum of money represented in any denomination may be represented in higher denominations by moving the decimal point to the left, one place for each denomination. A reduction is made to lower denominations by moving the decimal point to the right; thus,

$$6742 \text{ mills} = 67.42 \text{ dimes} = 6.742 \text{ dollars};$$
$$4671 \text{ dollars} = 46.71 \text{ cents} = 467.1 \text{ mills}.$$

142. We have already shown how to perform the operations of addition, subtraction, multiplication, and division of decimals; and the application of these rules to sums of money will require no further explanation, except to state that in cases of addition and subtraction care must be used in writing units of the same denomination in the same vertical column. This is not a necessity — only a convenience.

143. The coins in use are as follows:

Gold coins: the *dollar*, the *quarter-eagle*, the *half-eagle*, the *eagle*, and the *double-eagle*.

Silver coins: the *dollar*, the *half-dollar*, the *quarter-dollar*, and the *dime*.

Nickel coin: the *five-cent* piece.

Bronze coin: the *cent*.

The *mill* is used only in computation.

EXAMPLES XLVII.

Written Exercises.

1. Write the following in figures: two dollars thirteen cents, sixty dollars forty cents, three hundred dollars two cents, sixteen cents, six cents, three cents five mills.

2. Add \$14.15, \$37.24, \$156.50, \$.75, and \$1204.06.

3. Add \$2.04, 26.7 ct., 49.62 m., and 4.338 ct.

4. By how much is \$1507.45 greater than \$1429.78?

5. After spending \$145.45 a man had \$13.55 left; how much had he at first?

6. A man had originally \$1345.40. How much had he left after paying away \$135.25, \$416.67, and \$575.48?

7. What will 250 barrels of apples cost at \$2.75 per barrel?

8. A man bought 150 horses at \$125 each. He sold 50 at \$145 each and the rest at \$137.50 each. How much did he gain?

9. What is the value of 1400 bushels of wheat at 67 ct. a bushel?

10. A man bought 1250 bushels of oats at $38\frac{1}{2}$ ct. a bushel, and 1500 bushels of wheat at $65\frac{1}{2}$ ct. a bushel. What was the whole cost?

11. A man bought 15 pounds of cheese at \$.14 a pound, 9 pounds of coffee at \$.25 a pound, and 13 pounds of butter at 18 ct. a pound. How much did the whole cost?

12. Two men had between them \$1595, and one had \$155 more than the other. How much had each?

13. Multiply $684.93 by 6.75.

(Give answer to two decimal places, remembering that 5 or more mills increase the number of cents by one; anything under 5 mills is not considered.)

14. Multiply $71.41 by .23.

15. Divide $5687.98 by 27.3.

(Be sure in the answer to find out whether or not the mills will be as many as five.)

16. A man spent $4.86 in buying beef at $.09 per pound. How many pounds did he buy?

17. A man bought wheat at 64 ct. a bushel, and spent $736 altogether. How many bushels did he buy?

18. How many d. in $560.1?

19. How many e. in $270?

20. How many m. in $41.90?

21. How many ct. in 86420 m.?

22. How many $ in 86420 m.?

23. Divide 784 d. by 2.75, and write the answer as e.

24. Multiply $76 by .0025; of what denomination is the answer?

THE METRIC SYSTEM.

144. In almost all civilized countries, the United States and England being unfortunately exceptions, the different weights and measures have been arranged on the decimal system.

In France, Belgium, and Switzerland all sums of money are expressed in terms of the **Franc**, with its sub-unit the **Centime** ($\frac{1}{100}$ of a franc). In Italy, Spain, and Greece the standard unit of money is of exactly the same value as the franc, but is called by different names.

In Germany the standard unit is the **Mark**, with its sub-unit the **Pfennig** ($\frac{1}{100}$ of a mark).

In Austria the standard unit is the **Gulden**, with its sub-unit the **Kreutzer** ($\frac{1}{100}$ of a gulden).

In the United States the Metric System is used in scientific investigations and is authorized to be used in the Mint and Post Office.

145. In the **Metric System** of weights and measures, the fundamental unit is called a **Meter**. The *meter* is approximately the one ten-millionth part of the distance from the equator to the north pole. (A slight error was made in obtaining the meter, but its length remains as at first calculated.)

The standard units of *area*, *volume*, *capacity*, and *weight* are derived from the meter.

Decimal divisions of a standard unit are distinguished by the **Latin** prefixes *deci-*, *centi-*, *milli-*.

Decimal multiples of a standard unit are distinguished by the **Greek** prefixes *deka-*, *hekto-*, *kilo-*, *myria-*.

TABLES OF DECIMAL UNITS. — *Continued.*

146. Table of Linear Measures.

Length is distance in a straight line between two points.

The unit of linear measure is a meter.

10 millimeters (mm)	= 1 centimeter (cm).
10^{cm}	= 1 decimeter (dm).
10^{dm}	= 1 meter (m).
10^{m}	= 1 dekameter (Dm).
10^{Dm}	= 1 hektometer (Hm).
10^{Hm}	= 1 kilometer (Km).
10^{Km}	= 1 myriameter (Mm).

If the figure representing meters is put in units' place,

then	"	"	dm	"	"	tenths' place,
	"	"	cm	"	"	hundredths' place,
	"	"	Dm	"	"	tens' place,
	"	"	Hm	"	"	hundreds' place, etc.

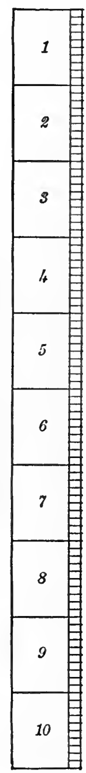
147. Length represented in any denomination may be represented in higher denominations by moving the decimal point to the left, *one* place for each denomination; a reduction is made to lower denominations by moving the decimal point to the right; thus,

$$14.45^{\text{m}} = 144.5^{\text{dm}} = 14450^{\text{mm}};$$
$$1256.4^{\text{cm}} = 12.564^{\text{m}} = 1.2564^{\text{Dm}};$$
$$2^{\text{Km}} = 200^{\text{Dm}} = 20000^{\text{dm}};$$
$$24.6^{\text{Hm}} = 2.46^{\text{Km}} = .246^{\text{Mm}}.$$

The same methods are used in operations here as in decimals [Arts. 29 and 49]. Units of the same denomination should be in the same vertical column.

The teacher should have a meter stick, properly graduated, and keep it constantly before the class. Not a word should be said about any other linear measure, known or unknown to the class.

1 decimeter = 10^{cm} = 100^{mm}.



148. Figures representing *decimal measures* of any kind are read just as figures representing integral and decimal numbers are read, and then the name of the denomination represented is read; thus,

14.45^{m} is read 'fourteen and forty-five hundredths meters,' (which means the same as if it were read 'one dekameter four meters four decimeters and five centimeters').

$.246^{\text{Km}}$ is read 'two hundred forty-six thousandths kilometers.'

EXAMPLES XLVIII.

Written Exercises.

1. Cut from cardboard a narrow strip, 1^{dm} long, and mark it accurately into tenths and hundredths.

2. Obtain the measure of the length of a book, and state the answer in dm and mm.

3. Mark your height on the wall, and obtain its measure in m; also in dm.

4. Measure a room in m; obtain length, breadth, and height.

5. Express 25^{m} as Dm; as Mm; as cm; as mm.

6. Write 126.73^{Dm} as m; as Km; as dm; as mm.

7. Add 14^{m}, 6^{dm}, 5027^{mm}, and 6.5^{Dm}. Answer in m.

8. Find the number of Dm in $12.62^{Mm} + 4267^{m} + 845^{cm}$.

9. How much longer is a room 12.65^{m} than a room 106^{dm} long?

10. Find $8469^{m} + 46892^{mm} - 468^{Dm} + 12^{dm} - 186^{cm}$ in m.

11. Multiply 78.6^{dm} by 125. Answer in m; also in Hm.

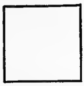
12. Four m of ribbon cost $16\frac{2}{3}$ ct. per m; find total cost [Theorem I, Art. 47; also Art. 50]. The answer is what fractional part of $1?

13. Divide 7469^{mm} by 11. Answer in three denominations.

149. Table of Surface Measures (Square Measures). That which has length and breadth, but no thickness, is called a **Surface**; thus,

The surface of a book has length and breadth.

A portion of a surface bounded by lines is called a **Figure.**



Square Centimeter.

A plane figure bounded by four equal sides, and whose four angles are equal, is called a **Square.**

Any square may be used as a unit of surface measure; for instance, a square centimeter, or a square meter.

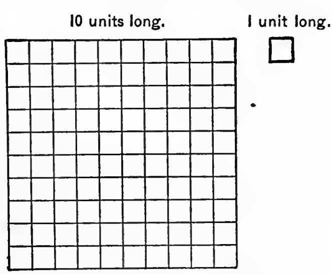
100 square millimeters (qmm)	= 1	sq. centimeter (qcm).
100^{qcm}	= 1	" decimeter (qdm).
100^{qdm}	= 1	" meter (qm).
100^{qm}	= 1	" dekameter (qDm).
100^{qDm}	= 1	" hektometer (qHm).
100^{qHm}	= 1	" kilometer (qKm).

FOR LAND SURVEYING.

1^{qm}	is called	a centar (ca).
1^{qDm}	" "	an ar (a).
1^{qHm}	" "	a hektar (Ha).

Sq. cm, etc., are often used instead of qcm, etc.

150. It is evident from the figure that, if one square is 10 times as long as another, its surface is 100 times as large; therefore,



A surface represented in any denomination may be represented in higher denominations by moving the decimal point to the left, *two* places for each denomination; a reduction is made to lower denomina-

tions by removing the decimal point to the right, *two* decimal places for each denomination; thus,

$15.6^{qm} = .156^{qDm}$; $10625^{qmm} = 1.0625^{qdm}$; $12^{a} = .12^{Ha}$.
$1.49^{qdm} = 149^{qcm}$; $1.0625^{qHm} = 10625^{qm}$; $1.6^{a} = 160^{ca}$.

15.6^{qm} is read 'fifteen and six-tenths square meters.'

EXAMPLES XLIX.

Written Exercises.

1. Cut from cardboard a piece to represent one qdm, and mark it accurately into qcm.

2. How many qcm in $\frac{1}{2}$ a qdm? How many in the square of $\frac{1}{2}$ a dm?

3. Mark out on the floor a qm. What would that be called if it were marked on the ground?

4. Write $30^{qm}\ 15^{qdm}\ 21^{qcm}$ as qm; as qHm; as qmm.

5. Express 31^{qm} as qcm; 14.1^{qm} as qdm; $.5^{qm}$ as qcm; 120.7^{qKm} as qm.

6. Read 15.14^{qDm} as qm; as qcm. Read 1.1^{qHm} as qDm; as qmm. Read 121^{a} as ca; as Ha.

7. Multiply 78.141^{qm} by 16, and answer in qmm.

8. Represent 15.6789^{qm} as qmm; 140^{qcm} as qHm.

9. In 1.49^{qdm} what might the .49 be called?

10. Read 15.6^{qm} as qDm and qdm.

11. Represent 1^{qKm}, 12^{qHm}, 1^{qm}, 4^{qmm} as qm.

12. Read, stating the number of units of each denomination represented, 167.08193^{qHm}.

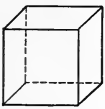
13. Divide 78965^{qDm} by 5, and answer in qm.

14. Add 167^{qm}, 200^{qKm}, 18.67^{qDm}, and 160003^{qmm}.

15. From 12.6^{a} subtract 4^{ca}.

16. A piece of ground containing 400^{a} is 2000^{dm} long; what is its breadth?

151. Table of Volume Measures (Cubic Measures).



Cubic Centimeter.

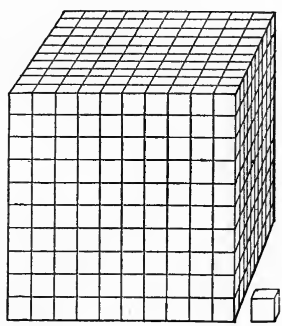
A solid bounded by six equal square surfaces is called a **Cube.**

Any cube may be used as a unit of cubic measure; for instance, a cubic centimeter or a cubic meter.

1000 cubic millimeters (cmm) = 1 cubic centimeter (ccm),
1000^{ccm} = 1 cubic decimeter (cdm),
1000^{cdm} = 1 cubic meter (cu m).

For measuring wood, 1 cubic meter is called a ster (st).

152. It is evident from the figure that, if one cube is 10 times as long as another, its volume is 1000 times as large; therefore,



10 units long.   1 unit long.

A volume represented in any denomination may be represented in higher denominations by moving the decimal point to the left, three places for each denomination; a reduction is made to lower denominations by moving the decimal point to the right, three places for every denomination.

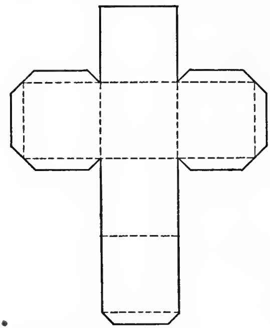
Thus, $4678^{cdm} = 4.678^{cu\,m}$; $8.67^{cu\,m} = 8670000^{ccm}$.

$4.678^{cu\,m}$ is read 'four and six hundred seventy-eight thousandths cubic meters,' which is equivalent to saying four cubic meters and six hundred seventy-eight cubic decimeters.

EXAMPLES L.

Written Exercises.

1. Cut from wood (or rubber, or cork) a piece to represent 1^{ccm}.



(Each student should carry in his pocket such a piece of wood, so that he can constantly refer to it.)

2. Cut from bristol board a piece shaped like the figure, having each of its six squares 1^{dm} long. Cut nearly through the cardboard in places represented by dotted lines, and make small flaps as shown. Such pieces can be made into cubes; a little mucilage on the flaps will keep the cubes in shape. Flaps should be out of sight in the finished cubes.

3. Mark the sides of the cube into qcm and learn how many ccm, like the one in your pocket, would be required to make a block as large as the cardboard cube.

4. How many ccm in $\frac{1}{2}$ a cdm? How many ccm in the cube of $\frac{1}{2}$ a dm; *i.e.*, in a cube 5^{cm} on an edge?

5. Write $105^{cu\,m}$ 215^{cdm} as cu m; 27^{cdm} as cu m.

6. Read $10.516^{cu\,m}$ as cdm; as cDm.

7. Read 10067^{cdm} as st.

8. Read $100601.41^{cu\,m}$, stating the number of units of each denomination represented.

9. Add 14.1^{cmm}, 14.1^{cdm}, and 14.1^{cDm}.

10. Divide 14.4^{ccm} by 12, and write the answer as ccm, and as cmm.

153. Table of Volume Measures (Liquid Measures).

The cubic decimeter is used as the unit of measure, and is called a **Liter.**

10 milliliters (ml)	= 1 centiliter (cl).
10^{cl}	= 1 deciliter (dl).
10^{dl}	= 1 liter (l).
10^{l}	= 1 dekaliter (Dl).
10^{Dl}	= 1 hektoliter (Hl).
10^{Hl}	= 1 kiloliter (Kl).

Comparing the above table with the one in Art. 151, we find that

$$1^{cdm} = 1^{l}.$$
$$1^{ccm} = 1^{ml}.$$
$$1^{cu\,m} = 1^{Kl}.$$

EXAMPLES LI.

Written Exercises.

1. Add 4.5^{l}, 2^{dl}, 47^{cl}, and 673^{ml}.

2. Express the answer to Ex. 1 in Kl, Hl, and cl.

3. How many liters of water in $4^{cu\,m}$?

4. Change 46.0949^{Kl} to l; to ml; to dl; to ccm.

5. Multiply $.678^{cl}$ by 2693; express the answer in cdm, and in cu m.

154. Table of Measures of Weight.

The attraction which the earth and any other body (on or off the earth) have for each other is called **Gravity.**

The amount of this attraction is called the **Weight** of the body.

The weight of 1^{ccm} of water is the unit of weight, and is called a **Gram.**

10 milligrams (mg)	= 1 centigram (cg).
10^{cg}	= 1 decigram (dg).
10^{dg}	= 1 gram (g).
10^{g}	= 1 dekagram (Dg).
10^{Dg}	= 1 hektogram (Hg).
10^{Hg}	= 1 kilogram (Kg).
10^{Kg}	= 1 myriagram (Mg).
10^{Mg}	= 1 quintal (Q).
10^{Q}	= 1 tonneau (T).

Observe, in the case of water, that

1^{ml} ($= 1^{ccm}$) weighs 1^{g};
1^{l} ($= 1^{cdm}$) weighs 1^{Kg};
1^{Kl} ($= 1^{cum}$) weighs 1^{T}.

155. **Kilogram** is called **Kilo.** Quintal is not often used.

The cubic centimeter of water, which is used as the standard unit, must be distilled, must be at a temperature of 39.2° F. (4° C.), and must be weighed in a vacuum at the level of the sea.

EXAMPLES LII.

Written Exercises.

1. Read 64.95^{g} as dg, cg, mg, and Mg.

2. Read 1256^{Hg} as Kg, Q, T, and g.

3. What is the weight of 1^{ml} of standard water? Of 10^{ml}? Of 1^{cl}? Of 10^{cl}? Of 3^{dl}? Of 3^{l}? Of 1000^{ccm}?

4. Iron is 7.8 times as heavy as water; what is the volume (in cdm) of 29.25^{Kg}? What is the weight of 2^{cdm}? Of 55^{ccm}? Of 7.2^{ccm}? Of 1.67^{ccm}? Of 125^{cum}?

5. Find the value (in grams) of $4^{Kg} - 18^{dg} + 18^{g} + 67.896^{mg} - 126.73^{cg} + 4^{T} - 11.6^{Mg}$.

6. Gold is 19.5 times as heavy as water; what is the weight of 1^{ccm}? Of one cubic meter?

CHAPTER VI.

NON-DECIMAL MEASURES.

156. The simplicity of calculations when using *decimal* measures is due to the facts that changes can be easily made from one denomination to another by moving the decimal point, and that several denominations can be expressed together in one set of figures.

In **Non-Decimal** measures, called also **Denominate** numbers and **Compound Quantities,** a variety of divisors is used in the different tables in order to change from low denominations to higher ones; also, it is unusual to express several denominations together in one set of figures.

For example, consider the case of the string mentioned in Art. 138. There, 12 inches equal 1 foot, and 3 feet equal 1 yard; and the length of the string must be expressed, not with the denominations together in one set of figures, but each denomination separately, — 6 yards, 1 foot, 6 inches.

To express a compound (Art. 139) quantity, express the number of units of each denomination separately, indicating the denominations, as in the above illustration.

To read compound quantities, read them exactly as expressed.

157. Table of Measures of Time.

The Standard Unit of Time is the **Mean Solar Day;** that is, the mean interval between two successive passages of the sun across the meridian of any place. A day is supposed to begin at midnight.

60 seconds (sec.)	= 1 minute (min.).
60 min.	= 1 hour (hr.).
24 hr.	= 1 day (da.).
7 da.	= 1 week (wk.).
365 da.	= 1 common year (yr.)
366 da.	= 1 leap year.

The year is divided into 12 months, called Calendar Months, which contain an unequal number of days, namely: January 31, February 28, March 31, April 30, May 31, June 30, July 31, August 31, September 30, October 31, November 30, and December 31.

Every fourth year contains 366 days, and is called Leap Year, and in these years February has 29 days. It is a Leap Year when the number of the year is exactly divisible by 4; thus, 1896 will be a Leap Year.

The *Solar Year* contains 365 da. 5 hr. 48 min. 46 sec., very nearly. Now it would clearly be very inconvenient to reckon by years which did not contain an exact number of days; hence, as the Solar Year contains *very nearly* $365\frac{1}{4}$ days, we have 3 years (called **Civil Years**) of 365 days each, and then one year of 366 days. The Solar Year is, however, somewhat *less* than $365\frac{1}{4}$ days, and the necessary correction is made by omitting three Leap Years in every 400 years, the years which are not counted as Leap Years (although divisible by 4) are the years which end the Centuries, and are such that the number of the Century is not divisible by 4. Thus, 1800 was not a Leap Year, and 1900 will not be a Leap Year; the year 2000 will, however, be a Leap Year.

158.* Reduction of Compound Quantities.

The method by which a compound quantity can be expressed as a simple quantity will be seen from the following example.

---

* The methods of reductions of compound quantities, also addition, etc., will be illustrated by the use of the above table because the different units are familiar to all.

Ex. *Reduce 7 da. 3 hr. 12 min. 26 sec. to seconds.*

```
   7 da.  3 hr.  12 min.  26 sec.
  24
 168
   3
 171 hr.                          7 da.                  =    168 hr.
  60         Adding the 3 hr.,    7 da. 3 hr.            =    171 hr.
10260                             171 hr.                =  10260 min.
   12        Adding 12 min.,      171 hr. 12 min.        =  10272 min.
10272 min.                        10272 min.             = 616320 sec.
   60        Adding 26 sec.,      10272 min. 26 sec.     = 616346 sec.
616320
    26
616346 sec.
```

159. To reduce a Simple Quantity to a Compound Quantity.

Ex. *Reduce* 14678 *sec. to hr., min., and sec.*

```
6,0)1467,8 sec.
  6,0)24,4 min. 38 sec.
       4 hr. 4 min. 38 sec.
```

Since 60 sec. make 1 min., if we divide the number of sec. by 60, we shall obtain the number of min. equivalent to 14678 sec., *i.e.*, 244 min., but shall have 38 sec. over. We then divide the number of min. by 60 and obtain the number of hours with 4 min. over.

160. Addition, Subtraction, Multiplication, and Division of Compound Quantities.

It will be seen that no new principle is involved. Care, however, must always be taken in regard to the number of units of one denomination required to make one unit of the next higher.

(*a*) **Compound Addition** [see Art. 142].

Ex. *Find the sum of* 14 *da.* 41 *min.* 11 *sec.*, 121 *da.* 18 *hr.* 16 *min.* 29 *sec.*, 201 *da.* 13 *hr.* 4 *sec.*, *and* 11 *hr.* 23 *min.* 30 *sec.*

da.	hr.	min.	sec.
14	0	41	11
121	18	16	29
201	13	0	4
	11	23	30
337	19	21	14

Here the sum of the seconds equals 74 = 1 min. 14 sec.; write the 14 and carry the 1. The number of min. = 81 = 1 hr. 21 min.; write the 21 and carry the 1. The number of hr. equals 43 = 1 da. 19 hr.; write the 19 and carry the 1. The number of days = 337.

(*b*) Compound Subtraction.

Ex. *From* 16 *da.* 12 *min. and* 50 *sec. subtract* 4 *da.* 12 *hr.* 13 *min. and* 54 *sec.*

da.	hr.	min.	sec.
16	0	12	50
4	12	13	54
11	11	58	56

Here 54 cannot be subtracted from 50; therefore we take 1 min. from the 12 min., change it to sec., and we have with the 50 sec. 110 sec. in all; subtract 54 sec. from 110 sec., and we have 56 sec. remainder. Now 13 from 11 we cannot take, therefore we take 1 hr. from the next column and proceed as before.

(*c*) Compound Multiplication.

CASE I. When the multiplier is not greater than 12.

Ex. *Multiply* 9 *da.* 10 *hr.* 31 *min.* 14 *sec. by* 7.

da.	hr.	min.	sec.
9	10	31	14
			7
66	1	38	38

Here 14 sec. × 7 = 98 sec. = 1 min. 38 sec.; write the 38 and carry 1. 31 min. × 7 = 217 min.; 217 min. + 1 min. = 218 min. = 3 hr. 38 min.; write the 38 and carry the 3. 10 hr. × 7 = 70 hr.; 70 hr. + 3 hr. = 73 hr. = 3 da. 1 hr.; write the 1 and carry the 3. Finally, 9 da. × 7 = 63 da.; 63 da. + 3 da. = 66 da. *Ans.* = 66 da. 1 hr. 38 min. 38 sec.

CASE II. When the multiplier can be seen to be the product of factors each not greater than 12.

Ex. *Multiply* 9 *da.* 10 *hr.* 31 *min.* 14 *sec. by* 35.

da.	hr.	min.	sec.
9	10	31	14
			7
66	1	38	38
			5
330	8	13	10

Case III. When the multiplier cannot be seen to be the product of factors each not greater than 12.

The following example will explain the method to be adopted, which will be seen to differ very little from the method adopted in the multiplication of simple quantities, the only apparent difference arising from the fact that we cannot at once write down the result of multiplying by 10, 100, etc.

Ex. *Multiply* 9 *da.* 10 *hr.* 31 *min.* 14 *sec. by* 257.

	da.	hr.	min.	sec.		
	9	10	31	14		
				10		
	94	9	12	20	= multiplicand	× 10
				10		
	943	20	3	20	= "	× 100
				2		
	1887	16	6	40	= "	× 200
2d line × 5	462	22	1	40	= "	× 50
1st " × 7	66	1	38	38	= "	× 7
	2416	15	46	58	= "	× 257

(*d*) Compound Division.

In division there are two cases to consider, according as the divisor is an abstract number or a concrete quantity of the same kind as the dividend [Art. 59].

Case I. To divide a compound quantity by an abstract number.

Ex. 1. *Divide* 22 *da.* 1 *hr.* 13 *min.* 1 *sec. by* 6.

	da.	hr.	min.	sec.
6)	22	1	13	1
	3	16	12	$10\frac{1}{6}$

Here, dividing 22 da. by 6, we have 3 da. with an undivided remainder of 4 da., which must be reduced to hr.; then we have 97 hr. in all to be divided by 6; the quotient equals 16 hr. with 1 hr. over. One hour and 13 min. = 73 min.; 73 min. ÷ 6 = 12 min. with 1 min. over. Finally, 1 min. = 60 sec.; 61 sec. ÷ 6 = $10\frac{1}{6}$ sec.

Ex. 2. *Divide* 9 *wk.* 6 *da.* 21 *hr.* 13 *sec. by* 33.

	wk.	da.	hr.	min.	sec.
3)	9	6	21	0	13
11)	3	2	7	0	$4\frac{1}{3}$
		2	2	49	$5\frac{28}{33}$

CASE II. When the divisor is a concrete quantity of the same nature as the dividend.

Ex. *Divide* 37 *da.* 20 *hr.* 6 *min.* 48 *sec. by* 12 *da.* 14 *hr.* 42 *min.* 16 *sec.* [Compare Art. 50.]

37 da. 20 hr. 6 min. 48 sec. = 3269208 sec.
12 da. 14 hr. 42 min. 16 sec. = 1089736 sec.
3269208 sec. ÷ 1089736 sec. = 3 (an abstract number).

(*e*) **To multiply or divide by a fraction.**

[Arts. 121 and 123.]

Ex. 1. *Multiply* 14 *da.* 2 *hr.* 12 *sec. by* $\frac{5}{7}$.

	da.	hr.	min.	sec.
	14	2	0	12
				5
7)	70	10	1	0
	10	1	25	$51\frac{3}{7}$

Ex. 2. *Divide* 14 *da.* 2 *hr.* 12 *sec. by* $\frac{5}{7}$.

	da.	hr.	min.	sec.
5)	14	2	0	12
	2	19	36	$2\frac{2}{5}$
				7
	19	17	12	$16\frac{4}{5}$

Both operations are understood because the nature of a fraction has been explained.

EXAMPLES LIII.

Written Exercises.

1. Add 17 da. 14 hr. 22 min. 12 sec., 13 da. 11 hr. 24 min. 18 sec., and 15 da. 33 min. 40 sec.

2. From 6 da. 12 sec. subtract 2 da. 4 hr. 12 min. 59 sec.

3. Multiply 7 da. 12 hr. 14 min. 25 sec. by 5.

4. Multiply 7 da. 12 hr. 14 min. 25 sec. by 18.

5. Multiply 7 da. 12 hr. 14 min. 25 sec. by 347.

6. A steamer makes a trip of 800 miles in 4 da. 8 hr. 12 min. 20 sec.; how many such trips could she make in 18 da. 10 hr. 52 min. 25 sec.?

7. Multiply 8 da. 5 hr. 8 min. 48 sec. by $\frac{3}{8}$.

MEASURES OF WEIGHT.

The units of measure for all weights are derived from the weight of a kernel of wheat taken from the middle of a ripe ear.

The name of such a weight is one **Grain** (gr.).

161. Table of Avoirdupois Weight.

The unit is a *pound* consisting of 7000 *grains*.

	16 drams (dr.)	= 1 ounce (oz.).
	16 oz.	= 1 pound (lb.).
	100 lb.	= 1 hundred-weight (cwt.).
	20 cwt.	= 1 ton (t.).
English	112 lb.	= 1 long hundred-weight.
	2240 lb.	= 1 " ton (l.t.).

Avoirdupois weight is used in weighing all ordinary substances.

The long ton is used in the Custom House, and in certain wholesale transactions.

The English Standard unit of weight is the **Imperial Pound** (Avoirdupois), and is the weight of a certain piece of platinum kept in the Exchequer Office.

EXAMPLES LIV.

Written Exercises.

1. Reduce 2 cwt. 15 lb. 12 oz. to dr.

2. How many gr. in an oz. avoirdupois?

3. Add 4 cwt. 72 lb. 14 oz. 11 dr., 34 lb. 12 oz. 2 dr., 8 cwt. 14 dr., 14 cwt. 56 lb. 3 oz., and 8 lb. 2 oz. 6 dr.

4. Reduce 1687649 dr. to units of higher denominations.

5. Reduce 16000 oz. to long tons, etc.

6. One boy weighs 125 lb. 10.5 oz.; how many such boys together weigh 22 cwt. 61 lb. 13 oz.?

162. Table of Troy Weight.

The unit is a *pound* consisting of 5760 *grains.*

24 grains (gr.)	= 1 pennyweight (pwt. or dwt.).
20 pwt.	= 1 ounce (oz.).
12 oz.	= 1 pound (lb.).

Troy weight is used in weighing gold and silver.

Diamonds and other jewels are spoken of as weighing so many carats. The carat is a little more than $3\frac{1}{6}$ grains.

The United States Standard unit of Weight is the **Troy Pound** (same as the English Pound Troy), and is the weight of a certain piece of brass in the custody of the Director of the U. S. Mint.

EXAMPLES LV.

Oral Exercises.

1. How many gr. in 3 pwt.? In $1\frac{1}{2}$ dwt.?
2. How many gr. in 2 oz.?
3. How many oz. in $3\frac{1}{4}$ lb.? In $5\frac{1}{6}$ lb.?
4. How many oz. in 70 pwt.? In 45 dwt.?
5. How many oz. in 480 gr.?
6. How many lb. in 78 oz.? In 43 oz.? In 400 pwt.?

Written Exercises.

7. Reduce 7563 dwt. to lb., etc.
8. Reduce 6 lb. 14 gr. to gr.

9. How many bronze cents weigh 1 lb., the 1 ct. piece weighing 48 gr.?

10. One hundred gold dollar pieces weigh 5 oz. 7 pwt. 12 gr.; what is the weight of one piece?

163. Table of Apothecaries' Weight.

The unit is a *pound* consisting of 5760 *grains*.

gr. 20 = 1 scruple (℈).
℈ 3 = 1 dram (ʒ).
ʒ 8 = 1 ounce (℥).
℥ 12 = 1 pound (℔.).

Apothecaries' weight is used by physicians when writing prescriptions and by druggists when *selling* drugs in *small* quantities. Avoirdupois weight is used by them when dealing in large quantities.

The symbols are always written at the left of the figures.

EXAMPLES LVI.

Oral Exercises.

1. How many gr. in ℈ 3? In ℥ 1? In ʒ $1\frac{1}{2}$?
2. How many ℥ in ℔. $2\frac{1}{2}$? In ℔. $3\frac{5}{6}$?
3. How many ℥ in gr. 480? In ℈ 48?
4. How many ℈ in ℥ 2? In gr. 60? In ℔. 1?
5. How many ℔. in ℥ 42? In ʒ 96? In ℥ 10 ʒ 16?

Written Exercises.

6. Reduce ℈ 7563 to ℔.
7. Reduce ℔. 4 ℥ 2 to gr.
8. Multiply ℔. 6 ℥ 7 ʒ 1 ℈ 2 gr. 15 by 16.
9. Divide ℔. 12 ℥ 3 ʒ 7 ℈ 2 gr. 4 by 4.

MEASURES OF LENGTH, SURFACE, AND VOLUME.

164. Table of Linear Measures.

The English Standard unit of Length is the **Imperial Yard** fixed by Act of Parliament to be the distance between two marks on a bar of metal kept in the Exchequer Office.

The U. S. Standard unit of Length is the same as that of England.

12 inches (in.)	= 1 foot (ft.).
3 ft.	= 1 yard (yd.).
$5\frac{1}{2}$ yd. / $16\frac{1}{2}$ ft.	= 1 rod (rd.).
320 rd.	= 1 mile (mi.).

FOR LAND SURVEYING.

7.92 inches	= 1 link (li.).
100 li. / 4 rd.	= 1 chain (ch.).
80 ch.	= 1 mi.

1 mi. = 1760 yd. = 5280 ft. = 1 statute mile.

Rods are sometimes called poles and perches. A furlong (fur.) = 40 rods = $\frac{1}{8}$ mi. Civil engineers use a chain 100 feet in length.

EXAMPLES LVII.

Oral Exercises.

1. Express 4 yd. as in.; 7 ft. as yd.
2. Reduce 18 ft. to rd., ft., and in.
3. How many in. in 2 yd. 1 ft.?
4. How many ft. in a surveyor's chain?
5. How many li. in 1 rd.?
6. How many yd. in 7 rd.?
7. Practise frequently the drawing (freehand) of a straight line 1 ft. long.

Written Exercises.

8. Reduce 40 rd. 6 ft. 7 in. to in.

9. Reduce 1 mi. to ft.

10. 803 in. to rd. and in.

165. Table of Square Measures.

The unit is any square, usually a square which is 1 ft. long.

144 square inches (sq. in.)	= 1 square foot (sq. ft.).
9 sq. ft.	= 1 square yard (sq. yd.)
$30\frac{1}{4}$ sq. yd. } $272\frac{1}{4}$ sq. ft. }	= 1 square rod (sq. rd.)
160 sq. rd.	= 1 acre (A.).
640 A.	= 1 square mile (sq. mi.).

1 A. = 160 sq. rd. = 4840 sq. yd. = 43560 sq. ft.

FOR LAND SURVEYING.

16 sq. rd. = 1 square chain (sq. ch.).
10 sq. ch. = 1 A.

Square measure is used for measuring land, flooring, and in fact, everything in which length and breadth have to be taken into account.

EXAMPLES LVIII.

Written Exercises.

1. Draw on the board (freehand) a figure representing a square foot, marking it accurately into square inches.

2. How many sq. in. in $\frac{1}{2}$ a sq. ft.? How many in the square of $\frac{1}{2}$ a ft.?

3. Having in mind a square 1 yd. long, how many sq. ft. in such a square?

4. Find by a figure the number of sq. yd. in 1 sq. rd.

5. Find by multiplication the number of sq. yd. in 1 sq. rd.

6. How many sq. in. in a square 2 ft. long? In a square 3 in. long?

7. Represent 3 A. 4 sq. rd. 50 sq. ft. as sq. ft.

8. Divide 58 A. 84 sq. rd. 3 sq. yd. 4 sq. ft. by 8.

166. Table of Cubic Measures.

The unit is any cube, generally a cube 1 in. long, or a cube 1 ft. long.

1728 cubic inches (cu. in.)	= 1 cubic foot (cu. ft.).
27 cu. ft.	= 1 cubic yard (cu. yd.).

FOR MEASURING WOOD.

16 cu. ft.	= 1 cord foot (cd. ft.).
8 cd. ft.	= 1 cord (cd.).

Cubic measure is used for measuring solid bodies in which length, breadth, and thickness have to be taken into account.

EXAMPLES LIX.

Written Exercises.

1. Make two cubes similar to the one in Ex. 2, p. 136, one cube an in. long, the other 4 in. long.

2. Mark the sides of the large cube into sq. in., and calculate how many cubes equal to the small cube might be cut from a block equal to the large cube.

3. Reduce 2 cu. yd. 1201 cu. in. to cu. in.

4. How many cu. in. in a cube 2 in. long? 5 in. long?

5. How many cords of wood in a pile containing 1541 cu. ft.? State answer to two decimal places.

6. Multiply 18 cu. yd. 9 cu. ft. 1063 cu. in. by 4.

167. Table of Liquid Measures.

The unit is a **Gallon** of 231 cu. in. (the old English wine gallon).

4 gills (gi.)	= 1 pint (pt.).
2 pt.	= 1 quart (qt.).
4 qt.	= 1 gallon (gal.).
$31\frac{1}{2}$ gal.	= 1 barrel (bbl.).
2 bbl.	= 1 hogshead (hhd.).

A gallon of water weighs 8.33 lb.
The quart is a volume of $57\frac{3}{4}$ cu. in.
The English Imperial Gallon contains 277.274 cu. in.

168. Table of Dry Measures.

The unit is a **Bushel** of 2150.42 cu. in. (the old English Winchester bushel).

2 pints (pt.)	= 1 quart (qt.).
8 qt.	= 1 peck (pk.).
4 pk.	= 1 bushel (bu.).

The quart is a volume of $67\frac{1}{5}$ cu. in.
The English Imperial bushel is 8 Imperial gallons = 2218.192 cu. in.

169. Table of Apothecaries' Fluid Measures.

60 minims, or drops (♏)	= 1 fluid dram (f℥).
fℨ8	= 1 fluid ounce (f℥).
f℥16	= 1 pint (O).
O8	= 1 gallon (Cong.).

EXAMPLES LX.

Oral Exercises.

1. How many pt. in 5 qt.? In 3 gal.?
2. How many pt. in 3 pk.? In 1 bu.?
3. Is the pt. in Ex. 1 equal to the pt. in Ex. 2?
4. Reduce 3 qt. 1 pt. to gi.; 1 bu. to qt.

5. How many ♏ in f℥1?

6. How many f℥ in ♏960?

Written Exercises.

7. Reduce Cong.1 to ♏.

8. Divide 14 bu. 3 pk. 5 qt. 1 pt. by 5.

9. Divide O 7 f℥ 10 fʒ 6 ♏ 59$\frac{3}{8}$ by $\frac{7}{8}$.

10. Reduce 750 dry qt. to liquid qt.

11. How many bushels of potatoes in a bin 4 ft. × 3 ft. × 2 ft.?

12. How many piles of 3 bu. each could be made out of a pile containing 8 cu. ft.?

13. Reduce 1 hhd. to cu. ft.

FOREIGN MONIES.

170. Table of English Money.

The unit is the **Pound** (£).

4 farthings (*far.*)	= 1 penny (*d.*).
12*d.*	= 1 shilling (*s.*).
20*s.*	= 1 pound (£).
21*s.*	= 1 guinea (*ga.*).

One farthing is written $\frac{1}{4}$*d.*; two farthings, or one half-penny, is written $\frac{1}{2}$*d.*; and three farthings is written $\frac{3}{4}$*d.* Thus, eightpence farthing is written 8$\frac{1}{4}$*d.*

The coins in use in England are as follows:

Gold coins: the sovereign (20*s.*) and half-sovereign (10*s.*).

Silver coins: crown (5*s.*), half-crown (2*s.* 6*d.*), florin (2*s.*), double florin (4*s.*), shilling, sixpence, and threepence.

Copper coins: penny, half-penny, and farthing.

171. German Money.

100 pfennigs (*pf.*) = 1 mark (*m.*).

172. French Money.

100 centimes ($c.$) = 1 franc ($f.$).

NOTE. See Art. 262 for equivalents in American money.

173. The value of a given fraction of a given concrete quantity is found as follows:

For example, to find $\frac{5}{18}$ of 3 lbs. 4 oz. Troy, we must divide 3 lbs. 4 oz. into 18 equal parts, and then take 5 of those parts; that is, we must divide by 18, and then multiply by 5. We may, however, first multiply by 5 and then divide by 18. Thus,

$$18\left\{\begin{array}{l|llll} & \text{lb.} & \text{oz.} & \text{pwt.} & \text{gr.} \\ 2 & 3 & 4 & & \\ \hline 9 & 1 & 8 & & \\ \hline & & 2 & 4 & 10\frac{2}{3} \\ & & & & 5 \\ \hline & & 11 & 2 & 5\frac{1}{3} \end{array}\right. \qquad \begin{array}{l|llll} & \text{lb.} & \text{oz.} & \text{pwt.} & \text{gr.} \\ & 3 & 4 & & \\ & & 5 & & \\ \hline 18\left\}\begin{array}{l}2\\9\end{array}\right. & \begin{array}{l}16\\ \hline 8\end{array} & \begin{array}{l}8\\ \hline 4\end{array} & & \\ \hline & & 11 & 2 & 5\frac{1}{3} \end{array}$$

174. The value of a given decimal of a given concrete quantity is found as follows:

Ex. *Find .54375 of 1 lb. Troy in lower denominations.*

$$\begin{array}{lr} & .54375 \\ & 12 \\ \hline \text{oz.} & = 6.525\cancel{0}\cancel{0} \\ & 20 \\ \hline \text{dwt.} & = 10.5\cancel{0}\cancel{0} \\ & 24 \\ \hline \text{gr.} & = 12.0 \end{array}$$

Here .54375 of 1 lb. = .54375 of 12 oz.,
which = 6.525 oz.;
.525 of 1 oz. = .525 of 20 pwt.,
which = 10.5 dwt.;
.5 of 1 dwt. = .5 of 24 gr.,
which = 12 gr.

∴ 6 oz. 10 dwt. 12 gr. = *Ans.*

EXAMPLES LXI.

Written Exercises.

1. Find $2\frac{2}{3}$ of 3 da. 12 hr.

2. Find $4\frac{5}{6}$ of 3 cwt. 36 lb.

3. Add $\frac{2}{3}$ of 2*s.* 6*d.*, $2\frac{4}{5}$ of 1*s.* 8*d.*, and $1\frac{1}{11}$ of 6*s.* 5*d.*

4. Find 1.625 of 1 da. − .02 of 1 wk.

5. Express 4.3125 lb. Troy in lb., oz., etc.

6. Find $.4\dot{3}\dot{6}$ of 1 mi.

7. By how much does $\frac{3}{5}$ of 1 mi. exceed $\frac{4}{7}$ of 310 rd. 1 yd.?

175. To express one quantity as a fraction of another, we proceed as follows:

Ex. 1. *Express* 14*s*. 6*d*. *as a fraction of* 15*s*. 8*d*.

$$14s.\ 6d. = 174d.$$
$$15s.\ 8d. = 188d.$$

Now
$$1d. = \tfrac{1}{188} \text{ of } 188d.$$
$$\therefore 174d. = \tfrac{174}{188} \text{ of } 188d.$$

Ex. 2. *Express* $2\frac{1}{5}$ *of* 1*s*. $7\frac{1}{2}d$. *as a fraction of* £1.

$$2\tfrac{1}{5} \text{ of } 1s.\ 7\tfrac{1}{2}d. = \tfrac{11}{5} \text{ of } \tfrac{39}{2}d. = \tfrac{429}{10}d.$$
$$£1 = 240d.$$

Now
$$1d. = \tfrac{1}{240} \text{ of } £1.$$
$$\therefore \tfrac{429}{10}d. = \tfrac{429}{10} \times \tfrac{1}{240} \text{ of } £1 = £\,\tfrac{429}{2400}$$
$$= £\,\tfrac{143}{800}.$$

176. To express one quantity as a decimal of another, we proceed as follows:

The method here is the reverse of that in Art. 174.

Ex. *Express* 10 *oz*. 11 *dwt*. 12 *gr*. *as the decimal of* 1 *lb*. *Troy*.

24	12 gr.
20	11.5
12	10.575
	.88125

Divide the grains by 24 to reduce to pwt.; add the 11 pwt. and divide 11.5 pwt. by 20 to reduce to oz; add the 10 oz. and divide 10.575 oz. by 12 to reduce to lb.

An excellent method is to express one quantity as a fraction of the other [Art. 175], and then reduce this common fraction to a decimal [Art. 130]; thus,

$$10 \text{ oz. } 11 \text{ dwt. } 12 \text{ gr.} = 5076 \text{ gr.}$$
$$1 \text{ lb.} = 5760 \text{ gr.}$$

Now, $\frac{5076}{5760} = .88125.$

EXAMPLES LXII.

Written Exercises.

1. Express 25 lb., 60 lb., 12 lb. 8 oz., and 6 lb. 4 oz. as fractions of 1 cwt.

2. What would be the measure of 4 yd. 2 ft. 8 in. if 1 yd. 1 ft. 7 in. were taken as the unit?

3. Express 1 oz. 6 dwt. 6 gr. as a decimal of 1 lb. Troy.

4. What decimal of an acre is 20 sq. rd. 5 sq. ft. 72 sq. in.?

5. Express £5. 12*s.* 6*d.* as a decimal of £10.

6. Express 2 mo. 7 da. as a decimal of 1 yr.

7. Express 7 mo. 12 da. as a decimal of 1 yr.

8. Express 10 mo. 15 da. as a decimal of 2 yr.

EXAMPLES LXIII.

Simple Examples in Reduction for Written Work.

Reduce:

1. 1 t. 3 cwt. 10 lb. to pounds.
2. 3 t. 12 cwt. 16 lb. to pounds.
3. 6 hr. 12 min. 10 sec. to seconds.
4. 12 hr. 5 min. 24 sec. to seconds.
5. 13 yd. 2 ft. 11 in. to inches.
6. 17 yd. 2 ft. 7 in. to inches.
7. 12 mi. 3 fur. 10 rd. to rods.
8. 13 mi. 5 fur. 26 rd. to rods.
9. 8 bu. 3 pk. 4 qt. to quarts.

10. 5 gal. 3 qt. 1 pt. to pints.

11. 5 A. 27 sq. rd. to square rods.

12. 17 A. 135 sq. rd. to square rods.

13. 13 sq. yd. 6 sq. ft. 100 sq. in. to inches.

14. 8 sq. yd. 7 sq. ft. 90 sq. in. to inches.

15. 6 lb. 7 oz. 10 dwt. 15 gr. to grains.

16. 18 lb. 9 oz. 15 dwt. 20 gr. to grains.

17. 3 wk. 5 da. 12 hr. to hours.

18. 16 da. 22 hr. 40 min. 35 sec. to seconds.

19. 12 t. 13 cwt. 75 lb. 7 oz. to ounces.

20. 5 t. 17 cwt. 68 lb. 14 oz. to ounces.

21. 2 mi. 3 fur. 80 yd. 2 ft. to feet.

22. 12 mi. 1200 yd. 1 ft. 7 in. to inches.

Reduce to tons, cwt., etc.:

23. 1462 lb. 25. 11597 lb. 27. 57812 oz.

24. 13574 lb. 26. 56214 oz. 28. 81974 dr.

Reduce to acres and square rods:

29. 315 sq. rd. 31. 1574 sq. rd.

30. 5142 sq. rd. 32. 3725 sq. rd.

Reduce to yards, feet, etc.:

33. 156 in. 34. 342 in. 35. 417 in. 36. 1179 in.

Reduce to lb., oz., dwt., gr.:

37. 517 dwt. 41. 13407 gr.

38. 574 dwt. 42. 24709 gr.

39. 3156 gr. 43. 35937 gr.

40. 4215 gr. 44. 51940 gr.

Reduce to bushels, pecks, etc.:

45. 156 pt. 46. 1472 pt. 47. 416 qt. 48. 1875 pt.

Reduce to square yards, etc.:

49. 1462 sq. in. 50. 2156 sq. in. 51. 3564 sq. in.

Reduce to days, hours, etc.:

52. 31572 sec. 53. 257672 sec. 54. 7142169 sec.

Calculate the number of

55. Sq. rd. in 1 sq. mi. 56. Sq. rd. in 1 A.

57. A. in 1 sq. mi.

58. Change 3.12 rd. to the decimal of a mi.

59. Change .2 sq. rd. to the fraction of an A.

60. Change ℔. $.0069\dot{4}$ to the fraction of a ʒ.

61. How many cd. of wood might be packed into a shed the size of your school-room?

177. The following cases are somewhat more difficult than those previously considered because one rod does not equal an exact number of yards.

Ex. 1. *Reduce* 31 *rd.* 3 *yd.* 2 *ft.* 11 *in. to inches.*

```
 31 rd. 3 yd. 2 ft. 11 in.
  5.5
 155
155
173.5 yd., the 3 yd. included.
    3
522.5 ft.,  "  2 ft.    "
   12
6281.0 in., " 11 in.    "
```

Here we have 31 rods to be multiplied by $5\frac{1}{2} = 5.5$. We might have multiplied by $\frac{11}{2}$.

CAUTION. In adding the yards or feet of the example while multiplying, care must be used in regard to the decimal point.

Ex. 2. *Reduce* 1885 *sq. rd.* 16 *sq. yd.* 6 *sq. ft. to sq. ft.*

```
    1885 sq. rd.
   30.25
57037.25 sq. yd., including the 16 sq. yd.
       9
513341.25 sq. ft.,    "     "   6 sq. ft.
```

Ex. 3. *Reduce* 6281 *in. to units of higher denominations.*

FIRST METHOD.

12)6281 in.		
3)523 ft.	5 in.	
5.5)174 yd.	1 ft.	
31 rd.	3.5 yd.	

5.5)174.0(31
165
90
55
35 [See Art. 68.]

	rd.	yd.	ft.	in.	
Integral part of answer	= 31	3	1	5	
Decimal " " " = .5 yd. =			1	6	
Sum =	31	3	2	11	*Ans.*

SECOND METHOD.

12)6281 in.	
3)523 ft.	5 in.
174 yd.	1 ft.
2	
11)348	
31 rd. . . . $\frac{7}{2}$ yd. [Art. 68.]	

It is shorter to multiply by $\frac{2}{11}$ than to divide by 5.5.

	rd.	yd.	ft.	in.	
Integral part	= 31	0	1	5	
Fractional part = $\frac{7}{2}$ yd. =		3	1	6	
Sum =	31	3	2	11	*Ans.*

Ex. 4. *Reduce* 513341.25 *sq. ft. to sq. rd., sq. yd., and sq. ft.*

9)513341.25		
30.25)57037	sq. yd.	8.25 sq. ft.
1885	sq. rd.	15.75 sq. yd.

	sq. rd.	sq. yd.	sq. ft.	sq. in.	
Integral part	= 1885	15	8		
Decimal part { .25 sq. ft. =				36	
Decimal part { .75 sq. yd. =			6	108	
Sum =	1885	16	6		*Ans.*

Here it is shorter and easier to divide by 30.25 than to divide by $\frac{121}{4}$ (*i.e.*, to multiply by $\frac{4}{121}$).

It will be noticed that only the integral part of any dividend is to be divided; the decimal part, if any, is to be regarded as a decimal part of the remainder.

178. In some cases of Reduction we cannot pass directly from one denomination to the other.

Ex. *How many lb. Troy are there in* 144 *lb. Avoir.?*

Since 1 lb. Avoir. = 7000 gr.,
144 lb. Avoir. = 7000 gr. × 144.

These grains are now reduced to lb. Troy in the usual manner.

MISCELLANEOUS MEASURES.

3 barleycorns	= 1 in.
4 in.	= 1 hand.
40 rd.	= 1 furlong.
1 geographical mi. = 6080 ft.	= 1 knot.*
3 knots	= 1 league.
6 ft.	= 1 fathom.

1 cu. ft. of pure water weighs 1000 oz. = $62\frac{1}{2}$ lb.

NUMBERS.

12 units	= 1 dozen.
12 dozen	= 1 gross.
12 gross	= 1 great gross.
20 units	= 1 score.

STATIONERY.

24 sheets	= 1 quire.
20 quires	= 1 ream.
2 reams	= 1 bundle.
5 bundles	= 1 bale.

EXAMPLES LXIV.

Written Exercises.

Reduce:

1. 10 rd. 2 yd. 1 ft. to feet.
2. 5 rd. 3 yd. 2 ft. to inches.
3. 1 mi. 3 fur. 20 rd. 1 yd. to yards.
4. 6 mi. 5 fur. 30 rd. 3 yd. to yards.
5. 18 mi. 11 rd. 3 yd. 1 ft. 6 in. to inches.
6. 27 mi. 273 rd. 2 yd. 2 ft. 7 in. to inches.
7. 6 mi. 52 yd. to yards.
8. 18 mi. 5 rd. 160 yd. 2 ft. 11 in. to inches.
9. 3 A. 16 sq. rd. to square yards.
10. 15 A. 24 sq. rd. to square yards.

---

* The knot recognized by the U. S. Coast and Geodetic Survey equals 6080.20 ft.

11. 3 A. 85 sq. rd. 16 sq. yd. 6 sq. ft. to square inches.

12. 16 sq. rd. 18 sq. yd. 5 sq. ft. 100 sq. in. to square inches.

Reduce to miles, etc.:

13. 6974 yards.

14. 21571 yards.

15. 15737 yards.

16. 6315 feet.

17. 51621 inches.

18. 158743 inches.

Reduce to acres, sq. rd., etc.:

19. 20812 sq. yd.

20. 38599 sq. yd.

21. 5172400 sq. in.

22. 8156179 sq. in.

Reduce:

23. 36 lb. Avoir. to lb. Troy.

24. 720 lb. Avoir. to lb. Troy.

25. 1 cwt. Avoir. to Troy weight.

26. 11 lb. 8 oz. Avoir. to Troy.

27. 350 oz. Troy to oz. Avoir.

28. 4 lb. 3 oz. 20 gr. to lb. and oz. Avoir.

29. 1 cwt. 9 lb. to ℔., ℥, etc.

30. ℔. 9 ℥ 6 ʒ 6 ℈ 2 gr. 5 to lb., etc., Avoir.

EXAMPLES LXV.

Written Exercises.

Add:

1.

da.	hr.	min.
5	17	42
3	11	53
7	19	37
11	7	21

2.

hr.	min.	sec.
1	41	15
6	17	39
7	35	42
5	16	13

3.

da.	hr.	min.	sec.
5	17	27	45
6	11	39	56
17	21	49	40
6	11	11	31

4.

cwt.	lb.	oz.
5	16	10
3	39	6
7	47	14
1	25	9

	lb.	oz.	dr.
5.	5	12	8
	4	13	12
	7	9	15
	3	11	14

	t.	cwt.	lb.	oz.
6.	5	15	17	3
	1	12	67	12
	15	17	20	11
	3	9	21	7

	lb.	oz.	dr.
7.	5	9	13
	7	14	12
	18	6	9
	3	11	11

	t.	cwt.	lb.	oz.
8.	16	17	19	14
	119	16	47	0
	72	12	37	13
	65	15	24	8

	cwt.	lb.	oz.
9.	6	24	10
	17	78	12
	14	7	14
	11	41	2

	lb.	oz.	dwt.
10.	6	4	19
	13	9	7
	2	11	17
	7	10	13

	oz.	dwt.	gr.
1.	1	17	23
	2	8	11
	5	15	7
	7	4	21

	lb.	oz.	dwt.	gr.
12.	5	11	16	18
	2	9	11	13
	7	10	15	21
	3	7	9	16

	yd.	ft.	in.
13.	5	2	9
	11	1	0
	13	2	7
	6	0	11

	yd.	ft.	in.
14.	16	1	7
	9	2	10
	20	0	8
	11	2	11

	yd.	ft.	in.
15.	15	0	9
	3	2	7
	18	1	11
	8	0	9

	mi.	rd.	yd.
16.	6	100	2
	3	140	4
	18	97	3
	2	15	2

	mi.	rd.	yd.	ft.	in.
17.	1	190	2	1	4
		3	3	0	11
	2	84	4	2	7
	3	180	3	1	9

	mi.	rd.	yd.	ft.	in.
18.	5	300	2	2	1
		15	3	1	9
	1	187	4	2	11
	2	74	5	0	9

	A.	sq. yd.
19.	5	12
	17	25
	3	18
	4	30

	A.	sq. yd.
20.	1	27
	16	19
	8	22
	19	7

	gal.	qt.	pt.
21.	5	2	1
	6	3	1
	4	1	0
	1	2	1

	gal.	qt.	pt.	gi.
22.	18	3	1	2
	4	1	0	3
	6	2	1	1
	1	1	1	1

	bu.	pk.	qt.	pt.
23.	3	2	5	1
	1	3	3	0
	10	0	6	1
	2	3	4	1

	℔.	℥	ʒ	℈	gr.
24.	4	10	6	2	5
	3	8	5	2	15
	1	0	1	1	6
	2	0	7	0	19

	Cong.	O.	f℥	fʒ
25.	1	6	12	4
	2	5	13	7
	1	2	3	5
	2	1	5	3

	℔.	℥	ʒ	℈
26.	1	11	7	2
	2	9	1	1
	3	6	6	0
	5	4	5	1

	cu. yd.	cu. ft.	cu. in.
27.	5.2	22.1	16.4
	1.3	19.2	126.9
	3.3	3.	14.3
	5.4	8.2	9.2

Answer in exact units.

EXAMPLES LXVI.

Written Exercises.

Subtract:

1. 5 da. 16 hr. 22 min. from 11 da. 18 hr. 10 min.

2. 15 da. 17 hr. 13 min. 42 sec. from 31 da. 9 hr. 11 min. 40 sec.

3. 5 cwt. 73 lb. 11 oz. from 7 cwt. 11 lb. 9 oz.

4. 6 lb. 10 oz. 11 dr. from 16 lb. 9 oz. 5 dr.

5. 7 t. 13 cwt. 15 lb. 12 oz. from 10 t. 11 cwt. 10 oz.

6. 3 lb. 4 oz. 10 dwt. from 9 lb. 1 oz. 5 dwt.

Find:

7. $\dfrac{73 \text{ lb. } 4 \text{ oz. } 10 \text{ pwt.}}{3} - 2(5 \text{ lb. } 10 \text{ oz. } 18 \text{ pwt. } 10.5 \text{ gr.})$.

8. 10 yd. – 5 yd. 1 ft. 10 in.

9. 29 yd. 1 ft. 4 in. – 17 yd. 2 ft. 11 in.

10. 17 mi. 1 fur. 150 yd. – 6 mi. 3 fur. 164 yd.

11. From ℔.4 ℥6 gr.17 subtract ℔.2 ℥7 ʒ3 gr.15.

12. From 18 sq. yd. 3 sq. ft. 17 sq. in. take 6 sq. yd. 7 sq. ft. 100 sq. in.

13. From 215 sq. yd. 3 sq. ft. 84 sq. in. take 118 sq. yd. 6 sq. ft. 112 sq. in.

14. From 25 A. take 15 A. 120 sq. rd. 10 sq. yd.

15. From 23 A. 40 sq. rd. 10 sq. yd. take 6 A. 125 sq. rd. 25 sq. yd.

16. Find 6 cu. yd. 24 cu. ft. 1200 cu. in. – 3 cu. yd. 25 cu. ft. 8 cu. in.

17.

bu.	pk.	qt.	pt.	gi.
7	1.2	0	1	3.6
3	2.4	1.3	0	2.4

Answer in exact units.

18.

gal.	qt.	pt.
10	1	
5	2	1

19.

O.	f℥	fʒ	♏
7	10	5	50
3	14	6	51

EXAMPLES LXVII.

Written Exercises.

Multiply:

1. 5 hr. 10 min. 33 sec., (i) by 5, (ii) by 7, (iii) by 9.

2. 5 cwt. 39 lb., (i) by 7, (ii) by 8, (iii) by 9.

3. 6 t. 17 cwt. 64 lb. 6 oz. 5 dr., (i) by 4, (ii) by 6, (iii) by 9.

4. 8 lb. 10 oz. 15 dwt. 20 gr., (i) by 5, (ii) by 7, (iii) by 12.

5. ℔ 6 ℥ 4.1 ʒ 2.3 ℈ 1.2 gr. 11 by 5.

6. 10 yd. 1 ft. 7 in., (i) by 8, (ii) by 11, (iii) by 12.

7. 8 mi. 215 yd., (i) by 5, (ii) by 8, (iii) by 12.

8. 1 mi. 20 rd. 4 yd., (i) by 7, (ii) by 56.

9. 15 sq. yd. 7 sq. ft. 100 sq. in., (i) by 6, (ii) by 11.

10. 4 cu. ft. 163 cu. in., (i) by 8, (ii) by 11.

11. 3 bu. 2 pk., (i) by 5, (ii) by 11.

12. 3 gal. 2 qt. 1 pt., (i) by 5, (ii) by 7.

13. 3 da. 17 hr. 10 min. 15 sec., (i) by 35, (ii) by 45.

14. 15 t. 12 cwt. 16 lb., (i) by 42, (ii) by 72.

15. 8 lb. 11 oz. 15 dwt. 18 gr., (i) by 49, (ii) by 84.

16. 3 yd. 2 ft. 10 in., (i) by 44, (ii) by 132.

17. 3 yd. 1 ft. 7 in. by 350.

18. 5 bu. 2 pk. by 420.

19. 12 da. 13 hr. 14 min. 12 sec. by 65.

20. 5 t. 7 cwt. 15 lb. by 94.

21. 3 lb. 4 oz. 12 dwt. 12 gr. by 124.

22. 3 cwt. 75 lb. 5 oz. by 257.

23. 15 sq. yd. 7 sq. ft. 82 sq. in. by 1212.

24. 6 t. 15 cwt. 7 lb. 3 oz. by 2341.

25. 2 lb. 4 oz. 16 dwt. 18 gr. by 3124.

26. 1 mi. 2 fur. 15 rd. 4 yd., (i) by 5, (ii) by 9.

EXAMPLES LXVIII.

Written Exercises.

Divide:

1. 22 da. 1 hr. 12 min. by 6.
2. 37 cwt. 3 lb. by 7.
3. 44 lb. 2 oz. 8 dr. by 8.
4. 52 lb. 10 oz. 13 dwt. by 9.
5. 153 yd. 2 ft. 1 in. by 11.
6. 95 A. 64 sq. rd. by 12.
7. 185 lb. 8 oz. 17 dwt. by 54.
8. 123 da. 10 hr. 45 min. by 50.
9. 1052 yd. 1 ft. by 132.
10. 251 A. 133 sq. rd. by 121.
11. 19 t. 14 cwt. 8 lb. 3 oz. 4 dr. by 500.
12. 214 t. 10 cwt. 44 lb. by 196.
13. 12 t. 3 cwt. 9 lb. by 37.
14. 309 t. 12 cwt. 14 lb. by 47.
15. 10 t. 6 cwt. 70 lb. 1 oz. by 57.
16. 37 yd. 2 ft. 3 in. by 151.
17. 35 t. 2 cwt. 63 lb. 2 oz. by 289.
18. 2237 bu. 1 pk. 7 qt. by 253.
19. 61 t. 1 cwt. 75 lb. by 2896.
20. 24 mi. 58 yd. 2 ft. 4 in. by 1234.
21. 36 mi. 4 fur. 23 rd. 3 yd. 1 ft. 6 in. by 10.
22. 55 mi. 7 fur. 26 rd. 1 yd. 1 ft. by 43.
23. 298 A. 39 sq. rd. 18 sq. yd. 2 sq. ft. 108 sq. in. by 73.

EXAMPLES LXIX.

Written Exercises.

1. Divide 2 tons 5 cwt. by 9 cwt.

2. Divide 6 oz. 10 dwt. by 13 dwt.

3. Divide 3 A. 50 sq. rd. by 19 sq. rd.

4. Divide 20 bu. 1 pk. by 2 bu. 1 pk.

5. How many pieces each 3 yd. 1 ft. long can be cut from a rope whose length is 180 yd.?

6. A wheel revolves once every 2 m. 15 sec.; how many times does it revolve in 1 hr. 48 m.?

7. The circumference of a tricycle wheel is 12 feet; how many times does the wheel turn round in a journey of 10 miles?

8. A field of 13 A. 80 sq. rd. is divided into allotments, each containing 1 A. 20 sq. rd.; how many allotments are there?

9. A man's average step is 2 ft. 11 in.; how many steps does he take in walking $3\frac{1}{2}$ miles?

10. How many jars, each containing 2 gal. 3 qt. 1 pt., can be filled out of a cask containing 46 gal.?

11. How many rails, each weighing 4 cwt. 37 lb., can be made out of 58 t. 19 cwt. 90 lb. of iron? What will each rail cost at 3 ct. a lb.?

12. How many times does 2 miles 76 yd. contain 14 yd. 1 ft. 6 in.?

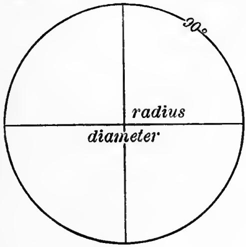
13. Each of a certain number of articles weighs 14 lb. 1 oz., and the total weight is 3 t. 75 lb.; how many are there?

14. How many times is 36 lb. 3 oz. 3 dwt. contained in 543 lb. 11 oz. 5 dwt.?

15. How many bullets, each weighing $2\frac{1}{2}$ oz., can be made from a quantity of lead weighing 7 cwt. 35 lb.?

16. A sovereign weighs 123 grains; how many can be made out of 3 lb. 5 oz. of standard gold?

179. Table of Circular Measures.



The plane figure whose bounding line is a curve everywhere equally distant from the centre is called a **Circle.**

The bounding line of a circle is called its **Circumference.**

Any part of a circumference is called an **Arc.**

If the circumference be divided into 360 equal parts, one of these parts is called an arc of one **Degree** (1°).

The unit is an arc of 1°.

60 seconds (″) = 1 minute (′).
60′ = 1 degree (°).
360° = 1 circumference (C.).

EXAMPLES LXX.

Written Exercises.

1. Add

5°	21′	15″
27°	41′	23″
196°	12′	39″
150°	2′	10″

2. From 182° 1′ 49″
Subtract 12° 50′ 50″

3. How many seconds in 90°?

4. How many degrees in 5678″?

5. How many circumferences in 1800°?

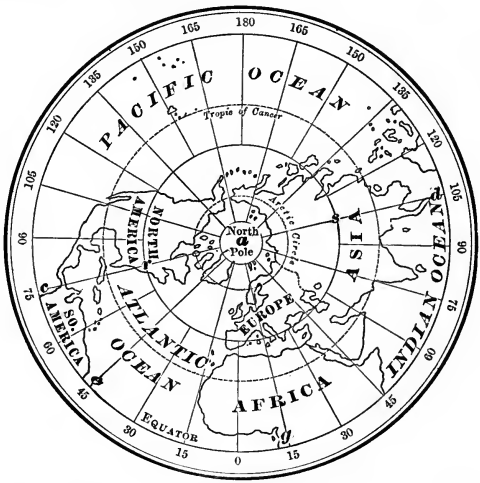
6. Reduce 100000″ to units of higher denominations.

180. Longitude and Time.

EXAMPLES LXX. — *Continued.*

Oral Exercises.

7. Let the figure represent a globe rotating on its axis; how many degrees does c move towards the present position of g while the globe is making $\frac{1}{6}$ of a rotation? $\frac{1}{9}$ of a rotation? $\frac{1}{24}$?



8. The earth is a rotating globe, and a point, as c or r, moves once around its circle in 24 hr.; how long does it take c to move to the present position of e, the arc ce being 30°? To the present position of g? Of d?

9. How long does it take r to move to the present position of s?

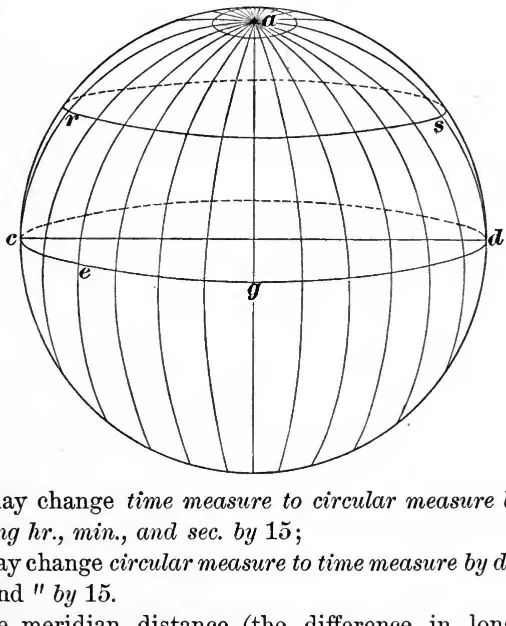
10. How long does it take the arc ac to reach the present position of the arc (meridian) ae?

11. How many degrees does the earth rotate in 1 hr.? In 1 min.?

12. How many arc minutes does the earth rotate in 1 min.? In 1 sec.?

13. How many arc seconds does the earth rotate in 1 sec.?

Since	15° rotation require 1 hr.
and	15′ rotation require 1 min.
and	15″ rotation require 1 sec.,



we may change *time measure to circular measure by multiplying hr., min., and sec. by* 15;
we may change *circular measure to time measure by dividing* °, ′, and ″ *by* 15.

The meridian distance (the difference in longitude) between two places is measured in units of circular measure, or in units of time measure.

181. Difference in Longitude, and in Time.

Longitude is reckoned either east or west from the meridian passing through Greenwich. It is evident that if two

places are either east of, or west from, Greenwich, the difference in longitude is found by subtraction; if one place is east and the other west, the difference is found by addition.

Ex. *Find difference in time between Cleveland*, 81° 40′ 30″ W., *and St. Paul*, 93° 4′ 55″ W.

$$\begin{array}{r} 93^\circ\ \ 4'\ 55'' \\ \underline{81^\circ\ 40'\ 30''} \\ 15\overline{)11^\circ\ 24'\ 25''} \\ 45 \text{ min. } 37 \text{ sec. } \textit{Ans.} \end{array}$$

EXAMPLES LXXI.

Written Exercises.

Find the difference in time between

1. Portland (Me.), 70° 15′ 40″ W., and Detroit, 82° 58′ W.

2. New York, 74° 0′ 3″ W., and Chicago, 87° 37′ 30″ W.

3. New York and Washington, 77° 2′ 48″ W.

4. Berlin, 13° 23′ 53″ E., and Paris, 2° 20′ 22.″5 E.

5. Berlin and New York.

6. Boston, 71° 3′ 30″ W., and San Francisco, 122° 24′ 15″ W.

7. Greenwich and Washington.

8. What is the longitude of St. Louis, the difference in time between New York and St. Louis being 1 hr. 5 min. 1 sec.?

9. The difference in time between Philadelphia and Chicago is 49 min. 50 sec.; what is the difference in longitude? What is the longitude of Philadelphia?

10. When it is 4 o'clock (p.m.) at Greenwich, what time is it at Washington?

11. When it is 1 o'clock (a.m.) at New York, what time is it at Berlin?

EXAMPLES LXXII.

Reduction of Metric Numbers to Non-Metric Numbers; also, of Non-Metric Numbers to Metric Numbers.

1. How many cm in 1 in.?
2. How many yd. in 17.6^{m}?
3. How many t. in 1^{Mg} of water?
4. How many sq. ft. in 1^{qm}?
5. How many cu. in. in 1^{l}?
6. How many lb. in 1$^{cu\,m}$ of water?
7. How many Mg in 1 l.t.?
8. How many ml in 1 qt. (liquid)?
9. How many g in ℔ 5?
10. How many gr. in 15^{cg}?
11. How many gr. in 500^{ccm} of water?
12. How many Hl in 5 pk.?
13. How many bu. in 3^{Kl}?
14. If either a qt. or a liter of milk cost 6 ct., which would you prefer to purchase?
15. Which would you prefer to buy, 1 A. or 2.5^{Ha} for the same money?
16. Find the value of ℥ 13 in g.
17. Find the value of 1^{Kg} in ℔.
18. Express 2 gal. 1 pt. 3 gi. as liters.
19. How many sters in 100 cu. ft.?
20. How many A. in 7^{Ha}?
21. Express $1\frac{1}{2}$ mi. as m and as Hm.
22. What cost 4 kilos of sugar at $5\frac{1}{2}$ ct. per lb.?
23. What costs $\frac{1}{2}$ a kilo of gold at \$1 a pwt.?

TABLES FOR CONVENIENT REFERENCE.

TIME.

60 sec.	= 1 min.
60 min.	= 1 hr.
24 hr.	= 1 da.
365 da.	= 1 yr.
366 da.	= 1 leap yr.

TROY WEIGHT.

24 gr.	= 1 pwt.
20 pwt.	= 1 oz.
12 oz.	= 1 lb.

AVOIRDUPOIS WEIGHT.

16 dr.	= 1 oz.
16 oz.	= 1 lb.
100 lb.	= 1 cwt.
20 cwt.	= 1 t.
112 lb.	= 1 l. cwt.
2240 lb.	= 1 l. t.

APOTHECARIES' WEIGHT.

gr. 20 = ℈ 1

℈ 3 = ʒ 1.

ʒ 8 = ℥ 1.

℥ 12 = ℔. 1.

LINEAR MEASURES.

12 in.	= 1 ft.
3 ft.	= 1 yd.
$5\frac{1}{2}$ yd. / $16\frac{1}{2}$ ft.	= 1 rd.
320 rd.	= 1 mi.
7.92 in.	= 1 li.
100 li.	= 1 ch.
80 ch.	= 1 mi.

SQUARE MEASURES.

144 sq. in.	= 1 sq. ft.
9 sq. ft.	= 1 sq. yd.
$30\frac{1}{4}$ sq. yd.	= 1 sq. rd.
160 sq. rd.	= 1A.
640 A.	= 1 sq. mi.
16 sq. rd.	= 1 sq. ch.
10 sq. ch.	= 1 A.

CUBIC MEASURES.

1728 cu. in.	= 1 cu. ft.
27 cu. ft.	= 1 cu. yd.
16 cu. ft.	= 1 cd. ft.
128 cu. ft.	= 1 cd.

LIQUID MEASURES.

4 gi.	= 1 pt.
2 pt.	= 1 qt.
4 qt.	= 1 gal.
$31\frac{1}{2}$ gal.	= 1 bbl.
2 bbl.	= 1 hhd.
1 qt.	= $57\frac{3}{4}$ cu. in.

DRY MEASURES.

2 pt.	= 1 qt.
8 qt.	= 1 pk.
4 pk.	= 1 bu.
1 qt.	= $67\frac{1}{5}$ cu. in.

APOTHECARIES' FLUID MEASURES.

♏60 = fʒ 1.

fʒ 8 = f℥ 1.

f℥ 16 = O 1.

O 8 = Cong. 1.

Synoptic Conversion of English and Metric Units.*

English to Metric.		Metric to English.	
1 in.	= **2.54** ^{cm}.	1^{m}	= **39.37** in.
1 yd.	= .9144 ^{m}.	1^{Km}	= 1093.61 yd.
1 mi.	= 1.60935 ^{Km}.	8^{Km}	= 5 mi. nearly.

1 sq. yd.	= .83613 ^{qm}.	$1^{qm} = 1^{ca}$	= 10.7639 sq. ft.
1 A.	= .404687 ^{Ha}.	1^{a}	= 119.599 sq. yd.
		1^{Ha}	= 2.471 A.

1 cu. in.	= 16.3872 ^{ccm}.	$1^{cu\,m}$	= 61023.4 cu. in.
1 cu. yd.	= .76456 $^{cu\,m}$.		= 35.3145 cu. ft.
1 qt. (U. S.)	= .94636 ^{l}.		= 1.30794 cu. yd.
		$\left.\begin{matrix}1^{cdm}\\1^{l}\end{matrix}\right\}$	= 61.023 cu. in.
			= .26417 gal. (U. S.)
			= 1.05668 qt. (U. S.).

1 gr.	= 64.7989 ^{mg}.	1^{g}	= 15.4324 gr.
1 lb. avoir.	= .45359 ^{Kg}.	1^{Kg}	= **2.20462** lb. avoir.
1 t. (2000 lb.)	= 907.18 ^{Kg}.	1^{T}	= 2204.62 lb. avoir.
1 l.t. (2240 lb.)	= 1.01605 ^{T}.	1^{T}	= .98421 l.t. (2240 lb.).

Weights.

1 bu. wheat	= 60 lbs.	1 stone	= 14 lbs.
1 " potatoes	= 60 "	1 bbl. pork	= 200 "
1 " beans	= 60 "	1 " flour	= 196 "
1 " corn	= 56 "	1 cental of grain	= 100 "
1 " barley	= 48 "	1 quintal of fish	= 100 "
1 " oats	= 32 "		

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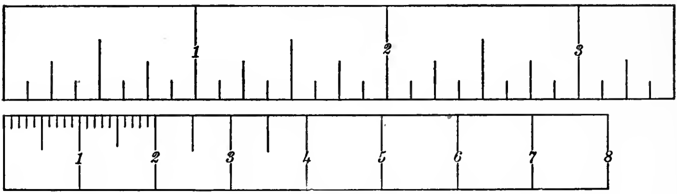
* Arranged from the Smithsonian Tables. Figures printed in black type should be memorized.

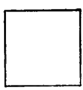
Inches.



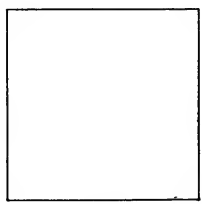
Centimeters.

1 in. = 2.54 cm.

1 qmm.

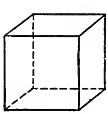


1 qcm.

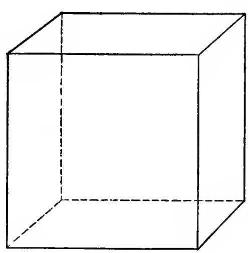


1 sq. in.

1 sq. in. = 6.45 qcm.



1 ccm.

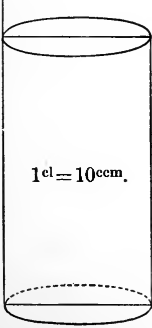


1 cu. in.

1 cu. in. = 16.387 ccm.

Diameter and height of a cylindrical liter measure and of a cylindrical quart measure.

$1^{l} = 61.063$ cu. in. $= 1000^{ccm}$.
1 qt. $= 57.75$ "



CHAPTER VII.

APPROXIMATION.

182. No continuous magnitude can be measured with *perfect* accuracy. When, for example, we endeavor to make two pieces of wire equally long, all that we can ensure is, that they shall be of the same length *so far as the eye, or other instrument, can judge;* however, they may, and probably will, differ by some thousandths or even hundredths of an inch.

In all questions involving continuous magnitude, such as length, weight, etc., we must, therefore, be content with *approximations* (more or less accurate) to the true measure. It follows that calculations dependent upon measurement can give only approximately accurate results.

For example, if we are told that a slab of stone is 17.6 inches long, and 12.4 inches wide, we are not to conclude that these are *perfectly accurate* measurements, but only that the measurements are near enough for *practical* purposes, the real length and breadth being at any rate less than 17.7 and 12.5 respectively.

If the given measurements were accurate, the area of the slab would be 17.6×12.4 square inches. The actual area may, however, have any value between 17.6×12.4 square inches and 17.7×12.5 square inches; that is, between 218.24 square inches and 221.25 square inches.

183. When the measure of any quantity is given, for example, as 3.628, it generally means that the measure is

not less than 3.628, and not greater than 3.629, the possible error made by stopping at the third decimal place being an error in defect less than one *one-thousandth* of the unit. Now, if the above measure had to be given as far only as hundredths of the unit, 3.63 would be more accurate than 3.62. This principle is often employed when approximate measures are given. Thus the quantity whose measure is 6.57684 would be most accurately given by 6.5768, 6.577, or 6.58 to four, three, or two decimal places respectively, the possible error in excess or defect being now not greater than *half* the unit represented by the last decimal place retained.

184. To find the sum of any numbers to any given number of decimal places, it would be necessary to consider the figures *two* places beyond, in order to see what had to be 'carried.'

Ex. 1. *Find, to* 3 *places of decimals, the sum of* 14.61825, 3.17924, .518479, *and* 154.017235.

$$\begin{array}{r|l}
14.618 & 25 \\
3.179 & 24 \\
.518 & 479 \\
\underline{154.017} & \underline{235} \\
172.333 &
\end{array}$$

Ex. 2. *Find, to within one one-thousandth of the whole, the sum of* 5.3184, 27.5162, 18.4196, and 23.0135.

$$\begin{array}{r|l}
5.31 & 84 \\
27.51 & 62 \\
18.41 & 96 \\
\underline{23.01} & \underline{35} \\
74.27 &
\end{array}$$

Here we have to find the sum correct to the first *four* figures. The sum of the numbers in the fifth column is 27, which is nearer to 30 than to 20. Hence, the most accurate sum to four figures will be 74.27.

185. The method of finding a product or a quotient to any required degree of accuracy will be seen from the following examples.

Ex. 1. *Find, to two places of decimals, the product of* 4.163 *and* 5.784.

$$\begin{array}{r|l|l} 4.1\cancel{6} & \cancel{3} & \\ & 5.7 & 84 \\ \hline 20.81 & 5 & \\ 2.91 & 4\,1 & \\ .33 & 3\,0 & \\ .01 & 6\,6 & \\ \hline 24.08 & & \end{array}$$

Arrange with the decimal point of the multiplier as above, and begin the multiplication from the left of the multiplier. The vertical line on the left gives the figures which are to be finally retained; it is, however, necessary to go two places beyond to see what should be 'carried' to the last column retained.

Multiply as usual so long as all the figures are to be retained. In the present case all the figures in the first two rows are to be retained.

Before multiplying by 8, cross out the last figure of the multiplicand, namely 3; then multiply 416 by 8, putting down the first figure of the product (adding in mentally what would be carried from the multiplication of the figure crossed out) in the last column. Now cross out another figure of the multiplicand, and multiply what remains by 4, again putting down the first figure of the product (with what must be carried from the multiplication of the last figure crossed out) in the last column. Proceed in this way to the end.

Since the sum of the figures in the fifth column is 18, the most accurate product we can give to two places of decimals is 24.08.

Ex. 2. *Find, to within one one-millionth of the whole, the product of* 51.6243 *and* 112.4167.

Here we have to find the product, correct to the first 7 figures.

$$\begin{array}{r|l|l} 51.6\cancel{2}\cancel{4} & \cancel{3} & \\ 11 & 2.4 & 167 \\ \hline 5162.43 & & \\ 516.243 & & \\ 130.248 & 6 & \\ 20.649 & 7\,2 & \\ .516 & 2\,4 & \\ .309 & 7\,4 & \\ .036 & 1\,3 & \\ \hline 5803.433 & & \end{array}$$

Ex. 3. *Find, to within one one-millionth, the quotient*

$$516.24175 \div 123.456.$$

$$\begin{array}{r}
12\cancel{3}.\cancel{4}\cancel{5}\cancel{6})516.24175(4.181585 \\
\underline{493\ 824} \\
22\ 4177 \\
\underline{12\ 3456} \\
10\ 07215 \\
\underline{9\ 87648} \\
19567 \\
\underline{12345} \\
7222 \\
\underline{6172} \\
1050 \\
\underline{987} \\
63 \\
\underline{61}
\end{array}$$

We have here to find the first seven figures of the quotient. Having found the first three figures in the ordinary way, the remaining four figures, being less by *two* than the number of figures in the divisor, can be found by a shortened process; namely, instead of annexing a naught at every stage on the right of the remainder as usual, we strike out the last figure on the right of the divisor instead, taking care, however, to use the last figure struck out to see what should be 'carried'.

Ex. 4. *Find, to the nearest penny, the value of*

$$£\,51.3125 \times 17.1874.$$

Since $\frac{1}{4}d. = £\,.001$ nearly, it will be unnecessary to retain more than four decimal places in the product.

Thus,

$$\begin{array}{r|l}
£\,51.3\cancel{1}\cancel{2}\cancel{5} & \\
\underline{17} & \underline{.1874} \\
513.125 & \\
359.1875 & \\
5.1312 & 5 \\
4.1050 & 0 \\
.3591 & 8 \\
\underline{205} & \underline{2} \\
£\,\mathbf{881}.9285 & \\
\underline{20} & \\
s.\ \mathbf{18}.5700 & \\
\underline{12} & \\
d.\ \mathbf{6}.84 &
\end{array}$$

Ans. £881. 18*s.* 7*d.*

EXAMPLES LXXIII.

Written Exercises.

Find the following to the nearest *thousandth* of the whole:

1. 14.625×31.857.
2. 15.816×19.714.
3. 156.423×175.45.
4. 138.714×89.47.
5. $314.2108 \div 18.306$.
6. $81.4623 \div 129.54$.
7. $15.8193 \times 6.7149 \div 1.3425$.
8. $115.416 \times 123.518 \div 119.417$.

Find, to within a millionth of the whole:

9. 198.4653×5.194238. 10. $8.10976429 \div 15.623$.

Find, to within one one-thousandth of the whole, the areas of the rectangles whose dimensions are:

11. 17.215 in. by 34.827 in.
12. 184.27 yd. by 112.53 yd.

13. Find, to 4 places of decimals:

(i) $$1+\frac{1}{1}+\frac{1}{1\times 2}+\frac{1}{1\times 2\times 3}+\frac{1}{1\times 2\times 3\times 4}+\cdots$$

(ii) $$1-\frac{1}{1}+\frac{1}{1\times 2}-\frac{1}{1\times 2\times 3}+\frac{1}{1\times 2\times 3\times 4}-\cdots$$

Find the value, to the nearest farthing, of

14. £31.625×12.8743. 15. £$119.48125 \times .46127$.

Find, to the nearest cent, the value of

16. \$$15.23 \times 18.24$.
17. \$$17.32 \times 112.428$.
18. \$$315.80 \times 175.297$.
19. \$$30.47 \times 2180.3079$.

EXAMPLES LXXIV.

Miscellaneous Examples, Chapters V, VI, VII.

Written Exercises.

1. Find $18 \times 19 \times 25 \times 16\frac{2}{3}$.

2. Express .035, .625, .12288 as common fractions in their lowest terms.

3. How many times is 14 yd. 1 ft. 6 in. contained in 244 yd. 3 in.?

4. Reduce 3 lb. 5 oz. 16 dwt. to gr., and express 1 oz. 16 dwt. 11 gr. in avoirdupois weight.

5. Find H.C.F. and L.C.M. of 936 and 2925.

6. Arrange $\frac{3}{62}$, $\frac{7}{144}$, and $\frac{4}{83}$ in order of magnitude.

7. Find the cost of 25 cwt. 25 lb. 12 oz. of a substance at $16 per cwt.

8. Find the value of 51 things, any four of which are worth £19. 3*s.* 1*d.*

---

9. Simplify $\frac{18}{17}(1 - \frac{64}{81}) + \frac{8}{11} \times \frac{1}{6}(\frac{1}{2} + \frac{5}{12})$.

10. What is the least number which must be added to 1000000 that the sum may be exactly divisible by 573?

11. Multiply 4 mi. 31 rd. $4\frac{1}{2}$ yd. by 3, and divide the result by 37.

12. The circumferences of the large and the small wheels of a bicycle are 143 in. and 40 in. respectively; how many more turns will the latter have made than the former in a distance of 13 mi.?

13. A man spends 7.75 francs a day; how much does he save in a year (of 365 days) out of a yearly income of 3000 francs?

14. A man spends 9.35 marks a day; how much in English money does he spend in a year (of 365 days), taking a mark to be worth $11\frac{3}{4}$*d.*?

15. A field is 192^m long and 57.75^m wide; how many Ha does it contain, and what would it cost at 7500 francs per Ha?

16. Reduce 772642 sq. yd. to A., sq. rd., and sq. yd.

17. Find, in hr., min., and sec., .6575 of a day.

18. What fraction of 8 lb. 11 oz. 2 dwt. 17 gr. is 10 lb. 9 oz. 16 dwt. 11 gr.?

19. Reduce $\frac{11}{16}$, $\frac{18}{125}$, and $\frac{602}{62500}$ to decimals.

20. A certain number was divided by 105, by 'short' divisions; the quotient was 192, the first remainder was 1, the second was 4, and the third was 6. What was the dividend?

21. Find by factors the square root of 23716.

---

22. What is the greatest sum of money of which both \$11.05 and \$188.50 are multiples?

23. How much would it cost to put gravel to a depth of a dm all over a court-yard 7.5^m by 5.75^m, the gravel and labor costing 8 francs per ster?

24. A grocer buys 15 cwt. of goods for \$24.50; at what rate per lb. must he sell to gain \$5.50?

25. A druggist buys 50 lb. of a certain drug; how many weeks will it last if he uses ℔ 1 ℥ 6 ʒ 1 ℈ 2 gr. 10 per week in putting up prescriptions?

26. Find $1\frac{2}{3}$ of 8 bu. 1 pk.

27. How many numbers, each 567, must be added that the sum may be greater than a million?

28. What is the greatest number of Sundays there can be in a year? On what day of the week will the first of February fall when the number of Sundays in a year of 365 days is greatest?

29. How many times can 3 yd. 1 ft. 7 in. be subtracted in succession from 115 yd. 2 ft. 11 in., and what will be the last remainder?

30. A bar of metal weighing 100 oz. 16 dwt. is made into coins, each weighing 1 oz. 8 dwt.; how many coins are made from the bar?

---

31. Simplify $1\frac{2}{3}$ of $\dfrac{6\frac{3}{4} - 2\frac{7}{9}}{3\frac{1}{2} - 1\frac{1}{18}} \div \left\{2\frac{5}{6} - \frac{3}{5} \text{ of } \dfrac{1}{4\frac{4}{5}}\right\}$.

32. A surveyor measured some ground and found it to be 10 ch. long and 4 ch. broad; how many A. were there?

33. What is the smallest number of exact acres that can have the form of a square?

34. What decimal of 1 mi. is 119 yd. 2 ft. 4 in.?

35. Find the value of 2 lb. 6 oz. 10 dwt. 12 gr. of gold at $216 per lb.

36. Find 105^2; $48 \times 33\frac{1}{3}$; $850 \div 16\frac{2}{3}$.

37. Express ℔. 1 as the decimal of 1 lb. Avoir.

38. Having given that a meter is 39.37 in., prove that the difference between 5 mi. and 8^{Km} is nearly 51 yd.

---

39. Add

$$\frac{4\frac{2}{3}}{6\frac{2}{3}} \text{ of } \frac{1\frac{1}{2} - \frac{5}{6}}{1\frac{1}{2} + \frac{5}{6}} \text{ of } \$4.65 \text{ to } \frac{6\frac{9}{11}}{11\frac{4}{11}} \text{ of } \frac{2\frac{1}{4} + 1\frac{1}{8}}{2\frac{1}{4} - 1\frac{1}{8}} \text{ of } \$1.15.$$

40. Find $\sqrt{\frac{5625}{9025}}$ and reduce the answer to lowest terms.

41. Express .88125 cwt. in lb. and oz.

42. Express 15 yd. 2 ft. 8 in. as the decimal of a mi.

43. Reduce 11.2765625 lb. to lb., oz., pwt., and gr.

44. Find to the nearest cent $48.96 × 72.8967.

45. Reduce 1000 sq. yd. to qm.

46. Reduce 1000^{1} to pt.

---

47. Express $.1\dot{3}\dot{6} \times 7.\dot{3} \div .4\dot{3}$ as a decimal.

48. Find the value of 43 sq. rd. $24\frac{1}{4}$ sq. yd. of building land at \$1815 per acre.

49. Find the greatest length of which both 1 mi. 4 fur. 16 rd. 2 yd. and 1 mi. 1 fur. 10 rd. 2 yd. are multiples.

50. Subtract $16\frac{1}{2} \times \dfrac{1\frac{3}{5}}{3 + \dfrac{1}{3\frac{1}{3}}}$ from $\dfrac{5\frac{2}{5} \text{ of } 7\frac{2}{9}}{8\frac{7}{24} - 3\frac{5}{12}}$.

51. Find $\sqrt{.004}$ to 4 decimal places.

52. Reduce 4^{Hg} to pounds Troy.

53. Simplify $\dfrac{7}{5 + \dfrac{3}{1 - \dfrac{1}{3 - \frac{5}{7}}}}$.

54. Find the annual cost of repairing a road 9 mi. 120 rd. 177 yd. long at \$88 per mi.

55. A vessel steams 18 knots an hour; to how many statute miles is this equivalent?

56. If a ccm of iron weighs 7.788^{g}, what will be the weight of a cu. ft.?

57. How many pieces each .17 in. long can be cut from a wire 21.09 in. long; and how long will be the piece left over?

58. Add .5125 of a yd., .62734 of a rd., and .018325 of a fur.; subtract the result from .0049 of a mi., and express the answer in yd., also in dm.

59. Find $\sqrt{4900546043.21156004}$.

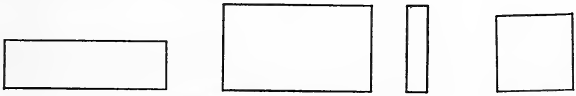
60. What is the least number which when
divided by 15 leaves a remainder 3, when
" " 18 " " " 6, "
" " 24 " " " 12?

CHAPTER VIII.

AREAS — VOLUMES.

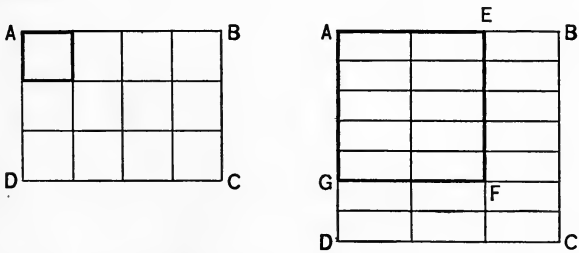
186. A plane figure [Art. 149] bounded by four straight lines, and whose four angles are equal, is called a **Rectangle.**

An equilateral *rectangle* is a **Square.**



Rectangles. Square.

The amount of surface included within the bounding lines of a figure is called its **Area**, and the area is measured by some square unit, — one sq. in., one sq. yd., or one qm, etc.



187. To find the Area of a Rectangle. — Let *ABCD* be the rectangle whose area is required.

Suppose, for example, that AB is 4 in. and that AD is 3 in. Divide AB into four equal parts and AD into three equal parts, and draw lines parallel to the sides as in the figure on the left.

Then the rectangle is divided into squares each of which is a sq. in.; and the number of these squares is clearly the product of the number of in. in AB by the number of in. in AD.

The above reasoning applies to all cases, both the length and the breadth of the rectangle being an integral number of in.

Now suppose, for example, that in the figure on the right AB is $\frac{3}{2}$ in., and that AD is $\frac{7}{5}$ in.

Let $AEFG$ be one sq. in. Divide AE into two equal parts, and AG into five equal parts, and GD into two equal parts. Then the subdivisions of AB will be all equal, as also those of AD. Hence, if lines be drawn as in the figure, $ABCD$ will be divided into 3×7 equal rectangles, such that the square inch $AEFG$ will contain 2×5 of these rectangles. Hence AB will contain $\frac{3 \times 7}{2 \times 5}$ square inches; that is, $(\frac{3}{2} \times \frac{7}{5})$ square inches.

From the above it follows that **the number of square inches (or square feet, etc.) in a rectangle is equal to the product of the number of inches (or feet, etc.) in the length by the number of inches (or feet, etc.) in the breadth.**

It should be noticed that the length and breadth must both be expressed in terms of *the same unit.*

For example, the area of a rectangle whose length is 2 ft. and breadth 6 in. is $(2 \times \frac{6}{12})$ sq. ft, or (24×6) sq. in.

The above rule for finding the area of a rectangle is often expressed shortly by the statement that **area = length × breadth.**

188. Now that we find the area of a rectangle, we can see that the relations between the different units given in the Table for Square Measure, on page 149, follow at once from the relations between the corresponding units in linear measure.

For, since 12 in. make 1 ft., (12×12) sq. in. make 1 sq. ft.
Since 3 ft. make 1 yd., (3×3) sq. ft. make 1 sq. yd.

Since $5\frac{1}{2}$ yd. make 1 rd., $(5\frac{1}{2} \times 5\frac{1}{2})$ sq. yd. make 1 sq. rd.

Again, 22 yd. make 1 ch., therefore (22×22) sq. yd. $= 484$ sq. yd. make 1 sq. ch.

Thus, 4840 sq. yd. = 10 sq. ch. = 1 A.

Also, 1 sq. mi. $= (1760 \times 1760)$ sq. yd. $= 1760 \times 1760 \div 4840$ A. $= 640$ A.

Ex. *Find the acreage of a rectangular field whose length is* 132 *yd. and whose breadth is* $38\frac{1}{2}$ *yd.*

The area $= (132 \times 38\frac{1}{2})$ sq. yd.
$= 5082$ sq. yd. $= \frac{5082}{4840}$ A.
$= \frac{21}{20}$ A. $= 1$ A. 8 sq. rd.

189. If the area of a rectangle be known, and also the length, the breadth can be at once found.

For example, to find the breadth of a rectangle whose length is 15 ft. and whose area is 200 sq. ft.

Since the product of the number of ft. in the breadth by the number of ft. in the length is equal to the number of sq. ft. in the area, we have

breadth $= (200 \div 15)$ ft. $= 13\frac{1}{3}$ ft. $= 13$ ft. 4 in.

EXAMPLES LXXV.

Written Exercises.

Find the areas of the rectangles whose lengths and breadths are as follows:

1. 14 ft., 12 ft.
2. 22 ft., 17 ft.
3. 25 yd., 17 yd.
4. 122 in., 114 in.
5. 5 ft., 17 in.
6. 10 yd., 23 ft.
7. 5 yd. 1 ft., 3 yd. 2 ft.
8. 21 yd. 2 ft., 18 yd.
9. 13 ft. 4 in., 9 ft. 2 in.
10. 11 ft. 9 in., 8 ft. 7 in.

Find the acreage of the rectangular fields whose lengths and breadths are as follows:

11. 319 yd., 275 yd.

12. 363 yd., 240 yd.

13. 400 yd., $214\frac{1}{2}$ yd.

14. $178\frac{3}{4}$ yd., $162\frac{4}{5}$ yd.

15. 550 yd., 400 yd.

16. 125 yd., $49\frac{1}{2}$ yd.

17. Length $\times$ breadth $= ?$

18. Area $\div$ length $= ?$

19. Area $\div$ breadth $= ?$

20. Find the area of a rectangular field whose length is 119.5^{m} and whose breadth is 96.2^{m}.

21. How many stones having rectangular tops $2^{\text{dm}} \times 1.2^{\text{dm}}$ will be required to pave a street 5^{Hm} long and 16.8^{m} wide, provided no spaces are left between the stones?

22. Find the area of a rectangular field whose length is 9 ch. 12 li. and whose breadth is 6 ch. 25 li.

23. Find the area of a rectangular field whose length is 9 ch. 25 li. and whose breadth is 7 ch. 75 li.

24. The area of a rectangle is 925 sq. in., and its breadth is 25 in.; what is its length?

25. What is the length of a rectangular table the area of whose top is 71 sq. ft. 16 sq. in., and the breadth 6 ft. 8 in.?

26. The area of a rectangular court-yard is 52 sq. yd. 2 sq. ft. 36 sq. in., and its length is 14 yd. 9 in.; what is its breadth?

27. What will it cost to paint the ceiling of a room whose length is 24 ft. 6 in. and breadth 16 ft. 6 in. at $.60 per sq. yd.?

28. What is the area of a square floor 7^{m} long?

29. What is the length of a square room whose area is 4225^{qdm}?

CARPETING, PAPERING, PLASTERING.

190. Examples like the following are of frequent occurrence:

Ex. 1. *How much will be the cost of a carpet for a room* 16 *ft.* × 20 *ft.* 3 *in. with carpet* 27 *in. wide at* $.75 *a yd., the strips running lengthwise?*

Number of strips = 20 ft. 3 in. ÷ 27 in. = 9.
Total length of carpet = 16 ft. × 9 = 144 ft. = 48 yd.
Cost = 48 yd. × $.75 = $36. [Art. 50.]

Ex. 2. *How much will be the cost of paper for the walls of a room* 19 *ft.* 3 *in. long,* 15 *ft.* 9 *in. wide, and* 12 *ft. high, the paper being* 21 *in. wide and costing* 5 *ct. per yard?*

Area of a wall = its length × its height.
∴ Area of 4 walls = distance around the room × height
= 70 ft. × 12 ft.
= 840 sq. ft.
Length of paper = 840 sq. ft. ÷ $\frac{21}{12}$ ft.
= 480 ft = 160 yd.
Cost of paper = 160 yd. × 5 ct.
= $8.00.

NOTE. In the preceding questions we have found the quantity of carpet (or wall paper) which would be required if it were of one uniform color throughout. When, as is almost invariably the case, there is a pattern on the carpet or paper, there must be a certain amount of waste, if the different lengths are properly fitted together. Moreover, wall papers are sold in lengths of 8 yards, called rolls; if, therefore, as in Ex. 2, 160 yards of paper were required, 20 rolls would have to be bought. American wall papers are generally 18 inches wide.

Ex. 3. *A room* 21 *ft. by* 19 *ft. has a Turkey carpet in it, a border* 3 *ft. wide all round being left uncovered by the carpet. The border was stained at a cost of* $.45 *a square yard, and the carpet cost* $4.50 *a square yard; what was the total cost?*

Since the border is 3 ft. wide all round the room, the length of the carpet must be 21 ft. − 3 ft. × 2 = 15 ft., and the breadth must be 19 ft. − 3 ft. × 2 = 13 ft.

Area of carpet $= 15 \text{ ft.} \times 13 \text{ ft.} = \frac{65}{3}$ sq. yd.
Price of carpet $= \$4.50 \times \frac{65}{3} = \$97.50$.
Border = area of room − area of carpet
$= (21 \times 19)$ sq. ft. $- (15 \times 13)$ sq. ft.
$= 204$ sq. ft. $= \frac{68}{3}$ sq. yd.
Cost of staining $= \$.45 \times \frac{68}{3} = \$10.20$.
Total cost $= \$97.50 + \$10.20$
$= \$107.70$.

EXAMPLES LXXVI.

Written Exercises.

1. How much carpet 27 in. wide will cover a room 22 ft. 6 in. long and 15 ft. 9 in. wide, carpet running lengthwise? What will be the cost at $1.20 per yd.?

2. A room is 8.3^{m} long and 5^{m} wide; how many meters of carpet must be purchased for such a room, the strips being 7^{dm} wide and running crosswise? How much in width must be turned under? In surface?

3. If you were carpeting a room $9^{m} \times 6^{m}$, which way would you have the strips run if they were 6.8^{dm} wide? How many less qm would be used than by running the strips the other way?

4. A room is 10 yd. 2 ft. long and 7 yd. 1 ft. 6 in. wide; find the cost of covering it with Turkey carpet at $1.25 a sq. yd.

5. Find the cost of carpeting a room $8\frac{1}{4}$ yd. long by 6 yd. 2 ft. broad with carpet $2\frac{1}{4}$ ft. wide at 84 ct. a yd.

6. What would be the expense of carpeting a room 24 ft. 6 in. by 18 ft. with carpet 27 in. wide, and which costs $1.20 a yd.?

7. How much carpet 27 in. wide would be required for a room 32 ft. by 23 ft., a margin 4 ft. wide being left uncovered?

8. Find the area of the four walls of a room 15 ft. long, 14 ft. wide, and 10 ft. high.

9. Find the area of the four walls of a room 16 ft. 4 in. long, 13 ft. 8 in. wide, and 11 ft. 4 in. high.

10. Find the area of the four walls of a room 10.5^{m} long, 5^{m} wide, and 4.9^{m} high.

11. Find the qm of the four walls of a room $7^{\text{m}} \times 4^{\text{m}} \times 3.2^{\text{m}}$, leaving out 3 windows, each $2^{\text{m}} \times 1.1^{\text{m}}$, and one door $2.4^{\text{m}} \times 1.3^{\text{m}}$.

12. Find the area of the four walls of a room 14 ft. 6 in. long, 13 ft. 10 in. wide, and 10 ft. 8 in. high.

13. A room is 18 ft. long, 13 ft. 6 in. wide, and 12 ft. high; how much paper 21 in. wide will be required to cover the walls, and what will be the cost at $.75 per piece of 12 yd.?

14. How much will it cost to paper a room 17 ft. 6 in. square, and 14 ft. 3 in. high, with paper 1 ft. 9 in. wide at 12 ct. a yd.?

15. A room is $6.1^{\text{m}} \times 5^{\text{m}} \times 4.2^{\text{m}}$; find the cost of plastering at $62\frac{1}{2}$ ct. per qm, allowing 7^{qm} for windows, door, and base-board. Do not forget the ceiling.

16. A room $8^{\text{m}} \times 5.6^{\text{m}} \times 4.2^{\text{m}}$ has 4 windows, each $2.1^{\text{m}} \times 1^{\text{m}}$, 2 doors, each $2.8^{\text{m}} \times 1.4^{\text{m}}$, and a base-board 2.4^{dm} high; find (i) the cost of plastering at 50 ct. per qm, (ii) the cost of paper 5^{dm} wide at $3.50 per roll of 10^{m}, (iii) the cost of a carpet 6.2^{dm} wide at $2 per m, all for this room. Find the total cost, allowing $15 for labor in putting on the paper and laying the carpet.

BOARD MEASURE.

191. A board which is one foot square and one inch or less in thickness has a measurement called one **Board Foot.**

Boards and squared timber are sold by the *Board Foot.*

192. The number of *board feet* in a board *one inch or less* in thickness is the same as the number of square feet in the surface.

The number of *board feet* in a stick of timber *more than one inch* thick is the number of square feet in the surface multiplied by the number of inches in the thickness.

Ex. 1. *How many board feet in a board* 20 *ft.* × 2 *ft.* × $\frac{7}{8}$ *of an inch?*

$$20 \text{ ft.} \times 2 \text{ ft.} = 40 \text{ board feet.}$$

Ex. 2. *How many board feet in a board* 15 *ft. long,* 18 *in. wide at one end and* 14 *in. wide at the other end, and* $\frac{1}{2}$ *an inch thick?*

$$15 \text{ ft.} \times 1\tfrac{1}{3} \text{ ft.} = 20 \text{ board ft.}$$

In this case the average width is used.

Ex. 3. *How many board feet in a stick of timber* 21.6 *ft. long,* 14 *in. wide, and* $3\frac{2}{3}$ *in. thick?*

$$21.6 \text{ ft.} \times 1\tfrac{1}{6} \text{ ft.} = 25.2 \text{ board ft.,}$$

if the stick were 1 in. or less in thickness. But we must multiply this result by $3\frac{2}{3}$, since the timber is $3\frac{2}{3}$ in. thick. Thus,

$$21.6 \text{ ft.} \times \tfrac{7}{6} \text{ ft.} \times \tfrac{11}{3} = 92.4 \text{ board ft.}$$

EXAMPLES LXXVII.

Written Exercises.

Find the number of board feet in the following:

1. A board 20 ft. × 2 ft. × $1\frac{1}{2}$ in.
2. A board 19 ft. 8 in. × 1 ft. 9 in. × $\frac{7}{8}$ in.

3. A timber 13 ft. × 1.1 ft. × $4\frac{1}{3}$ in.

4. A joist 11 ft. × 5 in. × 2 in.

5. A joist 16 ft. × 6 in. × $2\frac{1}{4}$ in.

6. Find the cost of each of the above five pieces at $20 per M; *i.e.*, by the thousand (board feet).

7. A hall 75 ft. × 50 ft. has two layers of boards for its floor, one kind costing $12 per M, and the other costing $21 per M; the floor timbers, 70 in number, are placed crosswise, and cost $16 per M. How much is the cost of material, the boards being $\frac{3}{4}$ in. thick, and the timbers 8 in. × 3 in.

DIMENSIONS OF CIRCLES.

193. Cut from cardboard a circular piece having a known radius, as 3^{cm}, or 3 in. Roll the circle (held upright) along a straight line and measure the line traversed in one complete rotation of the circle. This line will be found to be about $\mathbf{3\frac{1}{7}}$ times the diameter.

We have no means of finding the *exact* measure of the circumference in terms of the diameter, but by means of geometry we learn that the measure is **3.1416** (nearly) times the diameter. This is more exact than $3\frac{1}{7}$.

Hence, **Diameter × 3.1416 = Circumference,**

and **C ÷ 3.1416 = D.***

Using the results obtained in geometry for areas of circles, we have,

$$\text{Area} = R^2 \times 3.1416,$$
$$R^2 = \text{Area} \div 3.1416,$$
$$R = \sqrt{\text{Area} \div 3.1416}.$$

* *D*, *R*, and *C* stand for diameter, radius, and circumference, respectively.

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Ex. 1. *The diameter of a circle is* 10^{dm}; *find the circumference and area.*

$$C = D \times 3.1416 \qquad\qquad \text{Area} = R^2 \times 3.1416$$
$$= 10^{dm} \times 3.1416 \qquad\qquad = 25^{qdm} \times 3.1416$$
$$= 31.416^{dm}. \qquad\qquad = 78.54^{qdm}.$$

Ex. 2. *The area of a circle is* 50.2656 *sq. in.*; *find the radius and the circumference.*

$$R = \sqrt{\text{Area} \div 3.1416} \qquad\qquad C = D \times 3.1416$$
$$= \sqrt{50.2656 \div 3.1416} \qquad\qquad = 8 \times 3.1416$$
$$= \sqrt{16} \qquad\qquad = 25.1328 \text{ in.}$$
$$= 4 \text{ in.}$$

EXAMPLES LXXVIII.

Written Exercises.

Find the circumference when

1. $D = 14$ in.
3. $R = 15^{cm}$.
5. $D = 56$ yd.
2. $D = 75$ ft.
4. $R = 18^{m}$.
6. $R = \frac{1}{2}$ mi.

Find

7. R when $C = 314.16^{cm}$.
8. D when $C = 1$ mi.
9. D when $C = 153.9384$ yd.
10. R when $C = 47.124^{m}$.

Find area when

11. $R = 14$ ft.
13. $C = 37.6992^{dm}$.
12. $D = 20^{m}$.
14. $C = 251.328$ rd.

15. How many sq. ft. in the floor of a circular room whose diameter is 28 ft.?

16. The bottom of a round liter measure has a surface of 500^{qcm}; find the approximate radius.

17. Find the cost of concreting a circular fountain basin whose diameter is 20 ft., the work and material costing \$ 3.27 per sq. yd.

RECTANGULAR SOLIDS.

194. That which has length, breadth, and thickness is called a **Solid.**

A solid bounded by six rectangular [Art. 186] faces is called a **Rectangular Solid.**

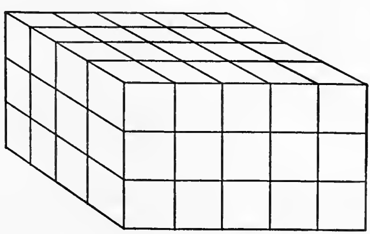
A cube [Art. 151] is one form of a rectangular solid.

Any substance (water, air, wood, etc.) may be a rectangular solid in form.

The space included between the bounding surfaces of a solid is called its **Capacity** (or **Volume**), and the capacity of a solid is measured by some cubic unit—one cu. in., one cu. ft., 1^{ccm}, or 1^{cum}, etc.

195. To find the Capacity of a Rectangular Solid.

Suppose, for example, that the dimensions of the solid are 5 in. by 4 in. by 3 in. We can divide the edges respectively into 5, 4, and 3 parts, each being one inch; and if planes be drawn through the points of division parallel to the outer faces of the solid, as in the figure, the whole solid will be divided into equal cubes each of which is a cubic inch.



There will be as many layers of cubes as there are inches in the height of the solid, and the number of cubes in each layer will be the product of the number of inches in the length by the number of inches in the breadth.

Thus, **the number of cubic inches (or cubic feet, etc.) in a rectangular solid is equal to the continued product of the number of inches (or feet, etc.) in its length, breadth, and thickness.**

196. Now that we can find the capacity of a rectangular solid, we can find the relations between the cubic yard, the cubic foot, and the cubic inch.

For $\quad 1 \text{ cu. yd.} = (3 \times 3 \times 3) \text{ cu. ft.},$
and $\quad 1 \text{ cu. ft.} = (12 \times 12 \times 12) \text{ cu. in.}$

Ex. 1. *Find the volume of a rectangular block of stone 12 ft. long, 7 ft. wide, and 1 ft. 6 in. high.*

Volume $= (12 \times 7 \times 1\tfrac{1}{2})$ cu. ft. $= 126$ cu. ft.

Ex. 2. *A beam 1 ft. 6 in. wide and 1 ft. 3 in. high contains $46\tfrac{1}{4}$ cubic feet of timber; what is its length?*

Since volume = length × breadth × thickness,

$$\text{length} = \frac{\text{volume}}{\text{breadth} \times \text{thickness}}.$$

$$\text{Hence, length required} = \frac{46\tfrac{1}{4}}{1\tfrac{1}{2} \times 1\tfrac{1}{4} \text{ ft.}} = 24\tfrac{2}{3} \text{ ft.}$$

Ex. 3. *How many gallons of water will a cistern hold if it is 6 ft. long, 4 ft. 6 in. wide, and 3 ft. 6 in. high?* [*A gallon contains* 231 *cu. in.*]

The cistern will hold

$$(72 \times 54 \times 42) \text{ cu. in.} = 163296 \text{ cu. in.}$$

Hence, the number of gallons required $= 163296 \div 231 = 706.\dot{9}\dot{0}.$

Ex. 4. *The external dimensions of a rectangular stone tank are: length 12 ft. 6 in., breadth 8 ft., and height 4 ft. The interior is also rectangular, and the sides and bottom are 3 in. thick. Find the number of cu. ft. of stone in the tank.*

The internal length $= 12 \text{ ft. } 6 \text{ in.} - 3 \text{ in.} \times 2 = 12 \text{ ft.},$
the internal breadth $= 8 \text{ ft.} - 3 \text{ in.} \times 2 = 7 \text{ ft. } 6 \text{ in.},$
and the internal height $= 4 \text{ ft.} - 3 \text{ in.} = 3 \text{ ft. } 9 \text{ in.}$

Now the volume of the stone is the difference between the volumes given by the external and internal dimensions.

Hence, volume required

$$= (12\tfrac{1}{2} \times 8 \times 4 - 12 \times 7\tfrac{1}{2} \times 3\tfrac{3}{4}) \text{ cu. ft.}$$
$$= (400 - 337\tfrac{1}{2}) \text{ cu. ft.} = 62\tfrac{1}{2} \text{ cu. ft.}$$

EXAMPLES LXXIX.

Written Exercises.

Find the volumes of the rectangular solids whose dimensions are

1. 5 ft. by 4 ft. by 2 ft.
2. 12 ft. by 6 ft. by 4 ft.
3. 3 yd. by $1\frac{1}{2}$ yd. by 2 ft.
4. 5 yd. by $2\frac{1}{2}$ yd. by 4 ft.
5. 6 ft. 4 in. by 4 ft. 3 in. by 2 ft. 6 in.
6. 7 ft. 9 in. by 5 ft. 3 in. by 3 ft. 6 in.
7. 5 yd. 1 ft. by 3 yd. 2 ft. by 2 ft. 9 in.
8. 6 yd. 9 in. by 2 yd. 1 ft. by 2 ft. 7 in.

9. A rectangular block of stone 4 ft. long and 2 ft. 6 in. broad contains $17\frac{1}{2}$ cu. ft. of stone; what is its height?

10. Find the length of a rectangular beam which contains 98 cu. ft. of timber and whose cross-section is 2 ft. square.

11. How many loads (cu. yd.) of gravel would be required to cover a path 150 yd. long and 4 ft. wide to a depth of 2 in.?

12. A school-room whose floor is 60 ft. by 40 ft. has accommodation for 360 children, allowing 100 cu. ft. of air for each child; what must be the height of the room?

13. If 1 gal. = 231 cu. in. and 1 gal. of water weighs 8.355 lb. Avoir., find the number of gal. and the weight of the water which would fall on an area of an A. during a rainfall of one in.

14. A tank is 21 ft. 4 in. long, 3 ft. wide, and 2 ft. deep; it is filled with water to within 3 in. of the top. What is the volume of the water, and what is its weight? [A cu. ft. of water weighs 1000 oz.]

15. What weight of water will fall on a road $\frac{1}{2}$ a mi. long and 30 ft. wide during a rainfall of an in.?

16. A level tract of land 20 mi. long and $\frac{3}{4}$ of a mi. broad is flooded to a depth of 4 ft. Given that a cu. ft. of water weighs 62.5 lb., find in t. the weight of the water on the land.

17. What is the capacity of a tank $20^m \times 8^m \times 2^m$? How many T of water will it hold? Reduce the T to t. (tonneaux to tons).

18. Find the total surface of the stone in Ex. **9**.

19. Find the inner surface of the tank in Ex. **17**.

20. A square room is 5^m long and 3^m high; how many cu. in. of air will the room contain?

21. A square room 9 ft. 3 in. high has a capacity of $1563\frac{1}{4}$ cu. ft.; what is the length of the room?

CYLINDERS.

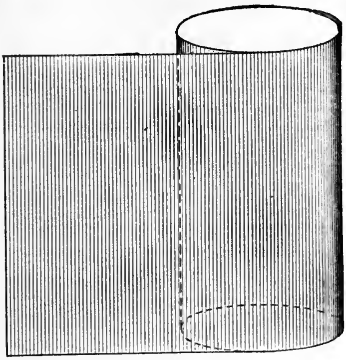
197. A solid whose ends are circles and whose curved surface is perpendicular to the ends is called a **Right Circular Cylinder.**

The ends are called **Bases.**

For example, a common lead pencil is a *right circular cylinder*; and some tin measures used for liquids are *right circular cylinders.*

NOTE. When cylinders are mentioned in this book, right circular cylinders are meant.

198. The total surface of a cylinder consists of two flat surfaces (circles) and a curved surface called the **Lateral Surface.**



If a piece of paper be fitted to a cylinder so as to cover all its *lateral surface* and then unrolled, it will be a rectangle whose length is the circumference of the cylinder and whose breadth is the height of the cylinder.

Hence, **lateral surface = C × height ;**

which (by Art. 193) $= D \times 3.1416 \times H$;

whence $H =$ lateral surface $\div (D \times 3.1416)$,

and $D =$ lateral surface $\div (3.1416 \times H)$.

EX. 1. *Find the total surface of a cylinder* 8 *in. high and the radius of whose base is* 2 *in.*

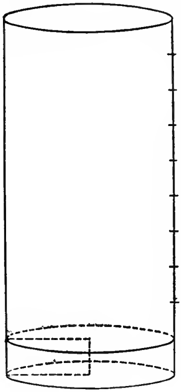
$$\begin{aligned} \text{Total surface} &= 2 \text{ bases} + \text{lateral surface} \\ &= 2\, R^2 \times 3.1416 + D \times 3.1416 \times H \\ &= 2\, R \times 3.1416\,(R + H) \\ &= 4 \times 3.1416 \times 10 \\ &= 125.664 \text{ sq. in.} \end{aligned}$$

NOTE. $H =$ height ; $D =$ diameter ; $C =$ circumference.

EX. 2. *The lateral surface of a cylinder is* 188.496^{qdm} ; *find the height when* $D = 6$.

$$\begin{aligned} H &= 138.496 \div (D \times 3.1416) \\ &= 188.496 \div 18.8496 \\ &= 10^{\text{dm}}. \end{aligned}$$

199. If a cylinder is 10 in. high, it is evident that it will contain 10 times as many cubic inches as if it were 1 in. high; since the number of cubic inches in a cylinder 1 in. high is the same as the number of square inches in the base, the



Volume of a cylinder = base × H,

or (Art. 193), $= \mathbf{R^2 \times 3.1416 \times H}$,

and $$H = \frac{\text{Volume}}{R^2 \times 3.1416},$$

and $$R^2 = \frac{V}{3.1416 \times H},$$

and $$R = \sqrt{\frac{V}{3.1416 \times H}}.$$

Ex. 1. *Find the volume of a cylinder whose height is* 10 *in. and the radius of whose base is* 6 *in.*

$$\begin{aligned} V &= R^2 \times 3.1416 \times H \\ &= 36 \times 3.1416 \times 10 \\ &= 1130.976 \text{ cu. in.} \end{aligned}$$

Ex. 2. *Find the radius of the base of a cylinder whose volume is* 125.664^{cdm} *and whose height is* 1^{m}.

$$\begin{aligned} R &= \sqrt{\frac{V}{3.1416 \times H}} \\ &= \sqrt{\frac{125.664}{3.1416 \times 10}} \\ &= 2^{\text{dm}}. \end{aligned}$$

EXAMPLES LXXX.

Written Exercises.

Find, in a cylinder, the

1. Lateral surface when $R = 3^{\text{dm}}$ and $H = 14^{\text{dm}}$.
2. Lateral surface when $R = 1$ in. and $H = 5$ in.

3. Total surface when $D = 10^{\text{m}}$ and $H = 8^{\text{m}}$.

4. H when lateral surface $= 1413.72^{\text{qdm}}$ and $D = 15^{\text{dm}}$.

5. V when $R = 3^{\text{dm}}$ and $H = 14^{\text{dm}}$.

6. V when $R = 1$ in. and $H = 5$ in.

7. V when $D = 10^{\text{m}}$ and $H = 8^{\text{m}}$.

8. H when $V = 125.664$ cu. ft. and $R = 2$ ft.

9. R when $V = 1570.8^{\text{l}}$ and $H = 20^{\text{dm}}$.

10. Measure in centimeters the height and diameter of some cylinder and calculate how many cubic centimeters of liquid it would hold if hollow. How many grams of water would it hold?

SPECIFIC GRAVITY.

200. Weigh accurately a stone. Then place it in a jar brimful of water and weigh the water which runs over. Now divide the weight of the stone by the weight of the water which ran over, and you will know how many times the weight of the stone is greater than the weight of the *same volume* of water.

The number of times that the weight of a substance is greater than the weight of the same volume of water is called the **Specific Gravity** (S.G.) of the substance.

A body floating in water displaces a weight of water equal to its own weight.

EXAMPLES LXXXI.

Written Exercises.

1. The S.G. of iron is 7.8; how much does a cu. ft. of iron weigh? A cu. in.? A ccm?

2. What is the weight of a cdm of silver, its S.G. being 10.5?

3. A rectangular iron tank weighs 25 kilos, and it floats on water; what is the weight of the water displaced? What is the volume of water displaced? What is the volume of iron in the tank?

4. A cubical liter measure weighs 150 grams; if put in water, what pressure must be added to its own weight to make it sink?

5. The S.G. of gold is 19.5; if a person can lift 125 lb., how many cu. in. of gold can he lift at one time?

6. What is the weight of a cdm of gold in pounds Avoir.? In pounds Troy? What is its value at $1 per pwt.?

If the stone mentioned in Art. 200 be weighed in air and then in water, the loss of weight will be found equal to the weight of the water which ran over. Therefore **if we divide the weight of a substance by its loss of weight in water, we shall obtain its S.G.**

7. A substance weighs 250 lb. in air and 125 lb. in water; what is its S.G.? How much water would run over if the substance were put into a jar brimful of water? What is the volume of the substance?

8. A piece of wood, S.G. .25, floats on water and displaces 40 ccm of water; what is the volume of the wood? How much iron must be attached to the wood to make it float under water?

9. How much weight will a ccm of iron lose when weighed in air and then in water?

NOTE. S.G. = weight in air ÷ loss in water.

10. A person weighing $146\frac{1}{2}$ lb. has a S.G. of $1.04\dot{1}\dot{7}$; how much does he weigh in water?

EXAMPLES LXXXII.

Miscellaneous Examples, Chapter VIII.

1. Find the cost of graveling a carriage drive 69 ft. 9 in. long and 16 ft. wide, at 30 ct. per sq. yd.

2. The outer and inner boundaries of a gravel path are squares, and the path is 4 ft. wide. The side of the square enclosed by the path is 50 yd. How much would it cost to gravel the path at $37\frac{1}{2}$ ct. a superficial yd.?

3. Find the cost of turfing a lawn tennis court which is 78 ft. long and 39 ft. wide, making a margin of grass 12 ft. wide at each end and 6 ft. wide at each side; the turf costing 4*d.* a sq. yd.

4. Find the prime numbers from 100 to 125 by using the sieve. [Art. 78.]

5. What will be the lowest cost of carpeting a room 33 ft. long and 24 ft. wide with carpet 27 in. wide and costing 85 ct. a yd., a border one yd. wide being left uncovered? How broad a strip must be cut off, or turned under?

6. Multiply 688.4 by 99; 460.01237 by 11.

7. Find the number of cu. ft. in a school-room by using its length, breadth, and height; find also the area of its six surfaces, including windows, etc.

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8. Find the cost of covering the floor of a hall 39 ft. $4\frac{1}{2}$ in. long by 20 ft. $11\frac{3}{4}$ in. wide with tiles each $5\frac{1}{4}$ in. by $4\frac{3}{4}$ in., and costing (including the labor of laying them) \$4.95 a hundred.

9. What is the weight of an iron girder 20 ft. long, and having 54 sq. in. sectional area, the weight of iron being 480 lbs. per cu. ft.?

10. A hall is 103.23 ft. long and 83.25 ft. broad, and it is to be paved with equal square tiles; what is the size of the largest tile which will exactly fit, and how many of them will be required?

11. A room 22 ft. 3 in. long, 17 ft. 9 in. wide, and 12 ft. 6 in. high has two windows each 5 ft. 3 in. by 3 ft. 4 in., a door 7 ft. by 3 ft. 9 in., and a fireplace 5 ft. 3 in. by 4 ft. 4 in. How many pieces (each 12 yd. long) of paper 21 in. wide would have to be bought to paper the room?

12. How many sq. ft. of boards are required for the floor of a circular hall 100 ft. in diameter?

13. What would be the weight of a beam of oak 5^{dem} square and 7^{m} long, on the supposition that the S.G. of oak is .895?

14. Divide 8921045 by 385 by the method of Art. 69.

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15. Supposing a postage stamp to be 1 in. long and $\frac{4}{5}$ of an in. broad, how many stamps would be required to cover a wall which is 15 ft. 6 in. long and 10 ft. 8 in. high?

16. What is the cost of paper for the hall of Ex. **12**, the hall being 25 ft. high, at \$.57 a roll, allowing 500 sq. ft. for windows, etc.?

17. A cubical cistern, open at the top, costs £16. 13*s.* 4*d.* to line with lead at 2*d.* per qdm; how many cu m of water will it hold?

18. I have a rectangular box cover $\frac{1}{2}$ of a meter long and $\frac{3}{25}$ of a meter broad to be painted in squares; what is the largest square I can use?

19. Find $\dfrac{\sqrt{3.0\dot{6}}}{2}$ to three decimal places.

20. Find the number of board feet in a stick of timber $8\frac{1}{2}$ in. square at one end, $8\frac{1}{2}$ in. by $5\frac{3}{4}$ in. at the other, and 21 ft. long.

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21. Find, by dividing by factors, $40579 \div 72$.

22. How many cdm of water are required to fill a cylindrical tank whose radius is 10^{dm} and whose height is 2^{m}?

23. A cylindrical tank holds 1884.96^{l}, and its height is $16\frac{2}{3}^{\text{dcm}}$; what is its radius?

24. How many cu. in. of wood are there in a wooden box whose external dimensions are 4 ft. 4 in. by 3 ft. 10 in. by 3 ft. 6 in., the wood being everywhere 1 in. thick?

25. An iron safe is everywhere $1\frac{1}{2}$ in. thick, and its external dimensions are 6 ft. by 4 ft. 6 in. by 3 ft. 6 in. How much does the iron weigh? [The S.G. of iron has been given several times.]

26. It cost 81.75*f.* to gravel a rectangular court-yard $8.25^{\text{m}} \times 4.8^{\text{m}}$ with gravel costing 7.5*f.* per st. What was the thickness of the layer of gravel?

27. An importer received 427 T of goods at 125*f.* per T; he paid the custom house $1602.92; his expenses of cartage, etc., were $212; for how much must he sell the goods per cwt. in order to make a profit of $2000?

28. The velocity of flow of water through a pipe 6^{cm} in diameter is 7.6^{dm} per sec.; how many l flow through in 11 sec.?

29. A room is 18 ft. 1 in. long, 11 ft. 8 in. wide, and 11 ft. 3 in. high; how many rolls of paper would be used in papering the walls, supposing that windows, etc., make up $\frac{1}{5}$ of the whole surface of the walls?

30. How much would it cost to carpet the room of Ex. **29**, the carpet being 1 yd. wide, at $1.66⅔ per yd.?

31. The S.G. of a piece of wood is .5; the wood being $8^{m} \times 8^{dm} \times 8^{cm}$, how many kilos would be required to sink the wood if placed on water?

32. Find $\sqrt{\left[2\,\{600 - (4\,\overline{150 - 25.5} - 16) + 3\} - (4.1\,\overline{12 - 2}) - \left(\frac{5\frac{2}{3}}{17} \times 90 + 50\right)\right]}.$

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33. The S.G. of gold is 19.5; find the weight of 1^{cdm} and the volume of 1 kilo.

34. A pile of wood is 8 ft. long, 4 ft. high, and 4 ft. broad; what is its volume in cu. ft.?

35. How many such volumes (Ex. **34**) in a pile 49.6 ft. long, 8 ft. high, and 4 ft. thick?

36. A quadrangle 120 ft. × 100 ft. has, in the center, a grass-plot 80 ft. × 60 ft.; find the cost of graveling the rest of it to a depth of 6 in. at 54 ct. a cu. yd.

37. At a certain place the annual rainfall was 24.15 in.; find the number of gal. which fell on each sq. mi.

38. A slate cistern open at the top is everywhere 1 in. thick, and the external dimensions are length 6 ft. 4 in., breadth 3 ft. 2 in., and height 4 ft. 8 in. Find the weight of the slate employed, assuming that 1 cu. ft. of slate weighs 2880 oz.

39. Find the number of gal. the cistern in the previous question will hold.

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40. What is the height of a cylindrical liter measure if the radius of its base is 5^{cm}?

Answer to the nearest tenth of a mm.

41. State the squares of 23, 34, 38, 47, 78, and 96, using the method indicated in Arts. 54 and 86.

42. Find by factors the H.C.F. and L.C.M. of 168, 2772, 4368, and 12474.

43. Find the weight of 12^{Hl} of alcohol, its S.G. being .81.

44. A room is 24 ft. 2 in. $\times$ 18 ft. 11 in.; which way would it be the cheaper to run the carpet strips, each strip being 27 in. wide?

45. Find the dividend when the divisor is $\frac{1.4}{87}$ and the quotient is $\frac{1.5}{.46}$.

CHAPTER IX.

RATIO — PROPORTION.

201. THE quotient of one number divided by another of the same kind may be called the **Ratio** of the first to the second.

Thus, the ratio of 6 ft. to 2 ft. is 3, and the ratio of 62 cwt. to 16 cwt. is $\frac{62}{16}$, or $3\frac{7}{8}$.

A *ratio* is expressed by the sign : placed between the two quantities; this sign means the same as the sign ÷.

Thus, $52^{m} : 26^{m}$ means $52^{m} \div 26^{m}$.

The first is read 'the ratio of 52^{m} to 26^{m}'; the second is read 'the quotient obtained by dividing 52^{m} by 26^{m}.' The answer is the same in both cases. A ratio may be also expressed in the form of a fraction; as, $\frac{52^{m}}{26^{m}}$.

202. The two quantities compared in a ratio are called the **terms** of the ratio.

The old names, **antecedent** and **consequent**, for the first and second terms respectively of a ratio, are still sometimes used.

The terms of a ratio must be of the same kind of magnitude; for we cannot compare, for example, tons with weeks, or acres with gallons.

When a ratio and one of its terms are given, the other term can at once be found.

Ex. 1. *A ratio is* 3, *and its first term is* 6*; find its second term.*

Here a dividend **6** and a quotient **3** are given; hence, the divisor $= 6 \div 3 = 2 =$ the second term of the ratio.

Ex. 2. *A ratio is* 4.5, *and its second term is* 10*; find its first term.*

Here a quotient **4.5** and a divisor **10** are given; hence, the dividend $= 4.5 \times 10 = 45 =$ the first term of the ratio.

Ex. 3. *What sum of money has to* \$30 *the ratio of* 5 : 8 ?

Here the ratio is $\frac{5}{8}$, and its second term is **\$30**; hence, $\$30 \times \frac{5}{8} = \$18.75 =$ the first term.

EXAMPLES LXXXIII.

Oral Exercises.

Find the indicated ratios, and in lowest terms:

1. 9 : 3; 16 : 4; 50 : 5; 20 : 9; 12 : 16.

2. 18 : 30; 75 : 100; 90 : 30; .5 : 5; 5 : .5.

3. 12 lb. : 5 lb.; 20 gr. : ʒ3; \$5 : \$55; \$.50 : \$5.50.

4. $24^{g} : 36^{g}$; $50^{cum} : 20^{cum}$.

5. A square 2 ft. long : a square 1 ft. long.

6. A cube 2 in. long : a cube 1 in. long.

7. A circle 6^{m} in diameter : a circle 2^{m} in diameter.

8. What is the ratio of a square to another square half as long? Twice as long? One-third as long? Three times as long?

9. What is the ratio of a cube to another cube half as long? Twice as long?

10. What is the ratio of a circle to another circle having twice the diameter? Three times the diameter? Five times?

Written Exercises.

Find, in lowest terms, the indicated ratios:

11. \$5 : \$7.50; \$3.25 : \$12.50; \$5.44 : \$39.10.

12. 3 t. 5 cwt. : 1 t. 15 cwt.; 3 cwt. 64 lb. : 4 cwt. 76 lb.

13. 7 mi. 208 rd. : 4 mi. 277 rd.; 2 oz. 11 pwt. 18 gr. : ℥1 ʒ2 ℈1 gr. 1.

14. Find what has to \$1.20 the ratio of 2 : 3.

15. Find what has to 1 da. 4 hr. 20 min. the ratio of 3 : 4.

16. Find what has to 11 cwt. 55 lb. the ratio of 7 : 11.

17. Find what has to £11. 14*s.* 9*d.* the ratio of 2.

18. Find the second term, the first term being 4 cd. 24 cu. ft., and the ratio being $\frac{8}{9}$, or 8 : 9.

PROPORTION.

203. When four quantities are such that the ratio of the first to the second is equal to the ratio of the third to the fourth, the four quantities are said to be **Proportionals.**

For example, the ratio \$5 : \$15 = the ratio 3 t. : 9 t. Hence, the four quantities, \$5, \$15, 3 t., and 9 t., are *proportionals.* The notation is as follows:

\$5 : \$15 = 3 t. : 9 t.;

or, \$5 : \$15 : : 3 t. : 9 t.

This is read, '\$5 is to \$15 as 3 t. is to 9 t.'; meaning that the ratio between \$5 and \$15 is the same as that between 3 t. and 9 t.

A *proportion* may be expressed in the form

$$\frac{\$5}{\$15} = \frac{3\text{ t.}}{9\text{ t.}}.$$

It follows that, when two fractions are equal, the terms of the fractions, taken in the order in which they are written, are proportionals.

Since the two terms of any ratio must be of the same kind of magnitude, the first and second terms of a proportion must be of the same kind, and the third and fourth terms must be of the same kind.

204.* The first and fourth, of four quantities in proportion, are called the **Extremes**, and the second and third are called the **Means**.

In the above case, the ratios are $\frac{5}{15}$ and $\frac{3}{9}$ respectively; and since $\frac{5}{15} = \frac{3}{9}$, it follows that $5 \times 9 = 15 \times 3$.

Thus, in this, and similarly in other cases, **the product of the extremes is equal to the product of the means.**

205. When any three terms of a proportion are known, the remaining term can be found.

Ex. 1. *What quantity* : 18 lb. : : \$4 : \$12 ?

For convenience, let x stand for the quantity to be found; then,

$$x \text{ lb.} : 18 \text{ lb.} :: \$4 : \$12,$$

whence [Art. 204] $12 \times x = 18 \times 4.$

$$\therefore \text{ once } x = 18 \times \tfrac{4}{12} = 6 \text{ lb.}$$

This is equivalent to saying that $\frac{4}{12}$ is a ratio, and 18 lb. is its second term; the first term (or dividend) must be $18 \times \frac{4}{12}$. [Art. 58; also 202, Ex. 2.]

Ex. 2. 27 *oz.* : 15 *oz.* : : *what quantity* : 25 *in.* ?

$$27 \text{ oz.} : 15 \text{ oz} :: x \text{ in.} : 25 \text{ in.}$$

$$15 \times x = 27 \times 25;$$

$$\text{once } x = \tfrac{27}{15} \times 25$$

$$= 45 \text{ in.} = \textit{Ans.}$$

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* It will be noticed that, in any proportion, when the first term is larger than the second, the third is larger than the fourth; that, when the first is smaller than the second, the third is smaller than the fourth.

We observe from the above that

$$\text{either extreme} = \frac{\text{product of means}}{\text{the other extreme}}.$$

Likewise,
$$\text{either mean} = \frac{\text{product of extremes}}{\text{the other mean}}.$$

206. When the second term equals the third term, we have but three different quantities in the proportion; the second is then called a **Mean Proportional** between the first and third, and the third is called a **Third Proportional** to the first and second.

Thus, in 4 : 6 : : 6 : 9, 6 is a mean proportional between 4 and 9, and 9 is a third proportional to 4 and 6.

Here $9 = \dfrac{\text{square of the second}}{\text{the first}}$, and $6 = \sqrt{\text{product of the extremes}}$.

EXAMPLES LXXXIV.

1. Find the fourth proportional to 4, 7, and 12.

2. Find the fourth proportional to 9, 8, and 3.

3. Find the fourth proportional to $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{5}{6}$.

4. Find the mean proportional between 4 and 9.

5. Find the mean proportional between 8 and 18.

6. Find the mean proportional between $\frac{2}{3}$ and $\frac{3}{2}$.

7. What quantity has to \$1.32 the same ratio that 4 ft. 3 in. has to 2 ft. 9 in.?

8. To what sum has 1*s.* 9*d.* the same ratio as 6 days 10 hr. has to 7 days 8 hr.?

9. Fill up the blank in the proportion

£1. 12*s.* 6*d.* : £2. 2*s.* 6*d.* : : 1 cwt. 18 lb. : ———.

Questions in which a missing term of a proportion has to be found, and other questions of a similar nature, are best treated by a method which we now proceed to consider.

THE UNITARY METHOD.

207. The method will be seen from the following examples:

Ex. 1. *If* 5 *lbs. of tea cost* \$2.75, *how much will* 8 *lbs. cost at the same rate?*

Since cost of 5 lbs. = \$2.75,
" " 1 lb. = \$2.75 ÷ 5.
∴ " " 8 lbs. = \$2.75 ÷ 5 × 8 = \$4.40.

Ex. 2. *If* 1 *cwt.* 24 *lb. of sugar cost* \$8.06, *how much will* 2 *cwt.* 46 *lb. cost at the same rate?*

Since cost of 1 cwt. 24 lb. (124 lb.) = \$8.06,
the " " 1 lb. = \$8.06 ÷ 124
= \$.065,
and " " 2 cwt. 46 lb. (246 lb.) = \$.065 × 246
= \$15.99.

Ex. 3. *How long would* 24 *horses take to consume the same quantity of food that* 45 *horses eat in* 16 *days?*

Since 45 horses eat the food in 16 days,
45 × 16 horses would eat the food in 1 day;
∴ 45 × 16 ÷ 24 " " " " 24 days.

Thus, the number of days required = 45 × 16 ÷ 24 = 30.

EXAMPLES LXXXV.

Oral Exercises.

1. If 5 t. cost \$35, what will 20 t. cost at the same rate?

2. A man walked 12 mi. in 3 hr.; how far would he walk in $1\frac{1}{4}$ hr.?

3. A certain quantity of food would be consumed by 18 persons in 15 da.; how long would it last 90 persons?

4. If 18 yd. are bought for $3.60, how much will 45 yd. cost?

5. How far should 15 t. be carried for the money charged for carrying 12 t. 5 mi.?

Written Exercises.

6. If 27 men can mow a field in 8 hr., how long will 36 men take to mow the same field?

7. If 18 yd. are bought for $16.50, find the price of 111 yd.

8. How far should 100 t. be carried for the money charged for carrying 75 t. a distance of 120 mi.?

9. If 19 men do a certain piece of work in 117 da., how long will it take 13 men to do the same work?

10. If 19 horses can be bought for $475, how many can be bought for $700 at the same rate?

11. If 25 cows cost $1387.50, how much will 6 cost at the same rate?

12. If a train runs 704 yd. in 12 sec., how long will it take to go half a mi.?

13. How many men would do in 20 da. the same amount of work as 15 men can do in 16 da.?

14. How long would 75 horses take to consume the same quantity of food that 40 horses eat in 15 da.?

15. If I lend a man $100 for 14 weeks, how long ought he to lend me $175 in return?

16. If 7 cwt. 4 lb. of steel cost $133.76, what will 3 cwt. 4 lb. cost at the same rate?

17. If 1 cwt. 19 lb. of coffee cost $41.65, how much will 5 cwt. cost at the same rate?

18. If 123 yd. of silk cost \$165.05, how much can be bought for \$58.05 at the same rate?

19. A man walks 9 mi. in 2 hr.; how long will he take to walk 12 mi. at the same rate?

20. If I lend a man \$350 for 34 weeks, how long ought he to lend me \$170 in return?

21. If gold is worth \$18.60 per oz., what is the value of a cup weighing 7 oz. 5 dwt. 12 gr.?

22. Find the value of 12 things any 7 of which are worth \$26.46.

23. If $3\frac{1}{2}$ lb. can be bought for \$5.46, how much can be bought for \$26.52?

24. If a t. of sugar cost \$110, how much will 8 cwt. 26 lb. cost at the same rate?

208. Each of the examples in the last exercise may be solved by the method of Art. 205.

For instance, in **7**, the price of 18 yd. holds the same ratio to the price of 111 yd. that 18 holds to 111; hence,

$$18 : 111 :: \$16.50 : x;$$
$$x = \$16.50 \times \tfrac{111}{18}$$
$$= \$101.75.$$

Here $\frac{18}{111}$ is the ratio and \$16.50 is the first term; therefore, we must divide \$16.50 by $\frac{18}{111}$ to find the second term.

Again, in **6**, the time required for 36 men is $\frac{27}{36}$ (ratio) of the time required for 27 men; therefore,

$$27 : 36 :: x : 8;$$
$$x = \tfrac{27}{36} \times 8$$
$$= 6 \text{ hours.}$$

EXAMPLES LXXXVI.

Written Exercises.

Perform examples 8–15 in the last exercise, using the method of Art. 208.

SIMILAR FIGURES — SIMILAR SOLIDS.

209. *Figures* or *Solids* which have the same shape are called **Similar Figures** or **Similar Solids.**

In the cases of rectangular figures, and rectangular solids, and cylinders [note, Art. 197], sameness in shape is determined by the ratios which exist between lines having the same relative positions. If these ratios are equal, the figures or solids have the same shape.

For instance, two rectangles, 12 ft. and 8 ft. in length, and 3 ft. and 2 ft. in height, are similar, because 12 ft. : 8 ft. : : 3 ft. : 2 ft.

Likewise, two cylinders, 15^{dm} and 9^{dm} in height, and 10^{cm} and 6^{cm} in diameter, are similar, because $15^{dm} : 9^{dm} :: 10^{cm} : 6^{m}$.

From preceding examples we have learned that *heights of squares are proportional to their lengths;* that *circumferences of circles are proportional to their diameters;* that *surfaces of squares or circles are proportional to the squares of lengths or diameters;* and that *volumes of cubes are proportional to cubes of lengths or heights or breadths.*

What is true of squares, circles, and cubes, is true of all similar figures; viz.,

(1) **Lines are proportional to lines** (height to length, etc.);

(2) **Surfaces are proportional to squares of corresponding lines;**

(3) **Volumes are proportional to cubes of corresponding lines.**

EXAMPLES LXXXVII.

Written Exercises.

1. The circumference of a circle is 12 in.; what is the circumference of a circle whose diameter is 3 times as great?

2. A rectangle $16^m \times 10^m$ is similar to another rectangle whose length is 4^m; what is the height of the second rectangle?

3. What is the area of a rectangle 5 in. long when a similar rectangle 9 in. long has an area of 32.4 sq. in.?

4. What are the comparative areas of two similar figures whose lengths are 8^{cm} and 17^{cm}?

5. Two cubes are 13^m and 1^m long; how large is the first in terms of the second?

6. Two similar cylinders have diameters of 5^{dm} and 3^{dm} respectively: compare their lateral surfaces; their bases; their volumes.

7. A cylindrical bin will hold 300 bu. of wheat; a similar one 3 times as high will hold how many bu.?

8. A cylinder 2^m high and 9^{dm} in diameter will hold how many kilos of water? What will be the diameter of a similar cylinder which will hold 10178784^g?

PROPORTIONAL PARTS.

210. Partnership.

When the ratio between the parts of a given quantity are known, the parts themselves can be at once found.

Ex. 1. *Divide* \$100 *between A and B so that A may have* \$3 *for every* \$2 *that B has.*

For every \$3 that A receives, B will receive \$2, *and the two together will receive* \$3 + \$2 = \$5.

Hence, A receives \$3 out of every \$5 of the whole;
∴ A " $\frac{3}{5}$ of the whole = $\frac{3}{5}$ of \$100 = \$60.
Also, B " $\frac{2}{5}$ of the whole = $\frac{2}{5}$ of \$100 = \$40.

Ex. 2. *The profits of a business are to be divided between the partners A, B, and C, so that A may have 4 parts, B 3 parts, and C 2 parts. How much does each get out of a profit of* \$4500?

If A has 4 parts to B's 3 parts and C's 2 parts, *A will have* 4 *parts out of* $(4 + 3 + 2)$ *parts divided between them all.*

Hence A will have $\dfrac{4}{4+3+2}$ of the whole;

$\therefore$ A will have $\frac{4}{9}$ of \$4500 = \$2000.
B " " $\frac{3}{9}$ of \$4500 = \$1500,
and C " " $\frac{2}{9}$ of \$4500 = \$1000.

Ex. 3. *Divide* \$23.50 *between A, B, and C, so that A's share may be to B's share as* 4 : 5, *and B's share to C's share as* 3 : 4.

Here A's share = $\frac{4}{5}$ of B's share, and B's share = $\frac{3}{4}$ of C's share;
$\therefore$ A's " = $\frac{4}{5}$ of $\frac{3}{4}$ of C's share = $\frac{3}{5}$ of C's share.

Hence A, B, and C have together $(\frac{3}{5} + \frac{3}{4} + 1)$ of C's share;

i.e., \$23.50 = $\dfrac{12 + 15 + 20}{20}$ " " "
= $\frac{47}{20}$ " " "

C's share = $\frac{20}{47}$ of \$23.50 = \$10; A's = $\frac{3}{5}$ of \$10; B's = $\frac{3}{5}$ of \$10.

Or thus:

A's share : B's = 4 : 5,
B's : C's = 3 : 4.

Now *multiply the terms of the two ratios by such numbers that the numbers corresponding to B's share may be the same in both.*

In the present case, multiply by 3 and 5 respectively. Then

A's share : B's : C's = 12 : 15 : 20.

Thus, A gets 12 parts out of $(12 + 15 + 20)$ parts altogether, etc.

Ex. 4. *Divide* £ 11. 12*s. between* 12 *men*, 8 *women, and* 20 *children, giving to each man twice as much as to each woman, and to each woman three times as much as to each child.*

A man's share = a woman's share × 2 = a child's share × 6.

Hence 12 men, 8 women, and 20 children will have

$(12 \times 6 + 8 \times 3 + 20)$ shares of a child.

Hence a child's share × $(72 + 24 + 20)$ = £ 11. 12*s.* = 232*s.*;

$\therefore$ a child's share = $\frac{232}{116}$*s.* = 2*s.*

Whence it follows that each man has 12*s.*, and each woman 6*s.*

Ex. 5. *Divide 532 into three parts proportional to $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$.*

Since $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$ are respectively $\frac{40}{60}$, $\frac{45}{60}$, and $\frac{48}{60}$, we have merely to divide into parts proportional to 40, 45, and 48.

Hence, as in Ex. 3, the parts are $\frac{40}{40+45+48}$ of the whole, etc.

EXAMPLES LXXXVIII.

Written Exercises.

1. Divide \$245 into parts in the ratio $3:4$.

2. Divide \$165 into parts in the ratio $2\frac{1}{2}:3$.

3. Divide \$33.15 into parts in the ratio $\frac{2}{3}:\frac{3}{4}$.

4. Divide \$54 into three parts proportional to the numbers, 1, 2, and 3.

5. Divide \$90.19 into parts proportional to the numbers, 7, 9, and 13.

6. Divide £17. 11*s.* into three parts proportional to 5, $5\frac{1}{2}$, and $7\frac{1}{2}$.

7. A sum is divided into parts proportional to the fractions, $\frac{2}{3}$, $\frac{4}{5}$, $\frac{7}{8}$; what fractional part of the whole is the first part?

8. The profits of a business are to be divided between the three partners in proportion to the numbers, 5, 3, and 2; how much does each receive out of a total profit of \$6237?

9. In a certain business A has 7 shares, B 5, C 3, and D 1 share. The profits are \$2410. Find each partner's share.

10. A provides \$5000, B \$3000, and C \$1250 to carry on a business. How much should each get out of a profit of \$555?

11. A, B, and C are partners in a business and have shares in proportion to the numbers, 4, 3, and 2, respectively, after $\frac{1}{20}$ per annum has been paid on the capital. The capital is $20000, of which sum A provided $12000, and B the remainder. How much does each receive out of a total yearly profit of $3400?

12. A, B, and C are partners in a business; C as manager receives $\frac{1}{10}$ of the net profits, the remainder being divided between A, B, and C in proportion to the numbers, 5, 4, and 3, respectively. In a certain year A's share of the profits amounted to $1520; what were the shares of B and C?

13.* The shares of A, B, and C of the capital in a business are as 4 to 3 to 2. After 4 months A withdraws half his capital, and the profits at the end of the year are $1518. How should this be divided between A, B, and C?

HINT. A has $\left\{\begin{array}{l}\text{4 for 4 mos.}\\ \text{2 for 8 mos.}\end{array}\right\} = \text{32 for 1 mo.}$

14. Divide $157.50 between A, B, C, and D, so that A may have as much as C and D together, B as much as A and C together, and D twice as much as C.

15. Divide $80 among 22 men, 26 women, and 82 boys, so that 2 men may have as much as 3 women, and 1 woman as much as 2 boys.

16. If 8 men can do as much as 14 women, and 5 women as much as 9 boys, divide $270 among 4 men, 6 women, and 9 boys in proportion to the work they do.

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* When partners put capital into a business for the same length of time, the case is one of Simple Partnership.

When capital is put into a business for different lengths of time, the case is one of Compound Partnership.

17. Divide \$1519.10 among three persons, A, B, and C, so that A may get one-fourth as much as B receives, and C may get one-tenth as much as A and B together.

18. Three partners, A, B, and C, had shares in a business proportional to the numbers, 4, 5, and 6, respectively. C retired and received as his share of the business \$15000. How much of this money should be paid by A and B respectively in order that after C's retirement their shares might be equal?

19. A and B, whose capitals were as 3 to 4, joined in business, and at the end of 4 months they withdrew $\frac{2}{5}$ and $\frac{3}{8}$ respectively of their capitals from the business. How should a gain of \$624 be divided between them at the end of the year?

20. The volumes of three substances contained in a certain mixture are proportional to the numbers, 2, 1, and 4, respectively; also the weights of equal volumes of the substances are as the numbers, 1, 32, and 16, respectively. Find the weight of the first substance contained in 3 lb. 1 oz. of the mixture.

MIXTURES.

211. The cost of a mixture of given quantities of two different ingredients is at once found when the prices of the separate ingredients are known.

Ex. 8 *lb. of tea costing* 30 *ct. per pound is mixed with* 3 *lb. of tea costing* 55 *ct. per pound; what is the cost of the mixture?*

The mixture cost 30 ct. $\times$ 8 + 55 ct. $\times$ 3 = 405 ct.

Hence, each pound of the mixture cost 405 ct. $\div$ 11 = $36\frac{9}{11}$ ct.

212. The ratio in which two different ingredients must be taken in order to make a mixture whose cost is any given sum *intermediate between the costs of the separate ingredients,* will be seen from the following examples.

Ex. 1. *In what ratio must tea costing* 30 *ct. per lb. be mixed with tea costing* 55 *ct. per lb. that the mixture may cost* 45 *ct. per lb.?*

The loss on the better quality is 10 ct. per lb.
The gain on the poorer quality is 15 ct. per lb.
The ratio between the loss and gain being $\frac{2}{3}$, we equalize loss and gain by making

$$\frac{\text{the number of lb. of the better quality}}{\text{the number of lb. of the poorer quality}} = \frac{3}{2}.$$

Ex. 2. *In what way must* 3 *kinds of tea worth* 30 *ct.*, 35 *ct.*, *and* 50 *ct. per lb. respectively, be mixed that the mixture may be worth* 38 *ct. per lb.?*

When there are 3 (or more) kinds of commodity, and only the price of the mixture fixed, there is an indefinite number of ways of satisfying the condition.

In the present case the gain on the lower two grades of tea, namely, 11 ct. on 2 lb. (1 lb. of each grade) must just balance the loss on the best grade, namely, 12 ct. per lb. The ratio between gain and loss $= \frac{11}{12}$. Hence, we must have 12 lb. of each of the lower grades and 11 lb. of the best grade.

Or, we may say that the gain on 2 lb. of 30 ct. tea with the gain on 1 lb. of 35 ct. tea (19 ct. in all) must just balance the loss (24 ct.) on a certain number of 2 lb. packages of 50 ct. tea. Here the ratio of gain to loss is $\frac{19}{24}$. Hence, we must have twenty-four 3 lb. packages (each package consisting of 2 lb. of 30 ct. tea and 1 lb. of 35 ct. tea) and nineteen 2 lb. packages of the 50 ct. tea.

Ans. = 48 lb., 24 lb., and 38 lb.

$$\text{Or,} \qquad \text{gain on} \left\{\begin{array}{l} 1\text{ lb. }30\text{ ct. tea} = 8\text{ ct.} \\ 4\text{ lb. }35\text{ ct. tea} = 12\text{ ct.} \end{array}\right\} = 20\text{ ct.};$$

$$\text{loss on} \quad 4\text{ lb. }50\text{ ct. tea} \qquad = 48\text{ ct.};$$

$$\therefore\ \text{gain} : \text{loss} :: 5 : 12.$$

Hence, we must have twelve 5 lb. packages (1 lb. of first kind with 4 lb. of second kind) and five 4 lb. packages of third kind.

Ans. = 12 lb., 48 lb., and 20 lb.

EXAMPLES LXXXIX.

Written Exercises.

1. What would be the cost per lb. of a mixture of 4 lb. of tea at 30 ct., and 6 lb. at 40 ct.?

2. What will be the cost of a mixture of 3 gal. of spirit at $2.80 per gal. and 5 gal. at $3.50 a gal.?

3. If 180 lb. of sugar which cost 4 ct. per lb. be mixed with 120 lb. which cost $5\frac{1}{4}$ ct. per lb., at what price must the mixture be sold so as to gain $\frac{1}{2}$ ct. per lb.

4. A milkman buys milk at 20 ct. per gal. He adds $\frac{1}{3}$ as much water as he buys milk, and sells the mixture at 28 ct. per gal. What is his gain per gal.?

5. In what ratio must two kinds of tea, which cost respectively 1*s.* 3*d.* and 1*s.* 9*d.* per pound, be mixed in order that the mixture may cost 1*s.* 5*d.* per pound?

6. In what ratio must biscuits worth respectively 11 ct. per lb. and 15 ct. per lb. be mixed that the mixture may be worth 12 ct. per lb.?

7. How much sugar worth $7\frac{7}{8}$ ct. per lb. must be mixed with 112 lb. of sugar worth $4\frac{1}{2}$ ct. per lb. in order that the mixture may be worth 7 ct. per lb.?

8. Tea at 66 ct. a lb. is mixed with tea at 78 ct. a lb. In what proportion must they be mixed, so that by selling the mixture at 77 ct. a lb. a profit of $\frac{1}{10}$ of the cost may be made?

WORK AND TIME.

213. We now consider problems with reference to work done in various times. These can all be solved by considering the fractional parts of the whole work which are done in a definite time.

Ex. 1. *One man can mow a field in* 30 *hr., and another man can mow the field in* 60 *hr.; how long would it take them working together to do it?*

The first man mows $\frac{1}{30}$ of the whole in 1 hr.,
the second man mows $\frac{1}{60}$ of the whole in 1 hr.;
therefore, two together mow $\frac{1}{30}+\frac{1}{60}$ of the whole in 1 hr.

And, as the two together would mow $(\frac{1}{30}+\frac{1}{60})=\frac{1}{20}$ of the whole in 1 hr., they would mow the whole in 20 hr.

Ex. 2. *A cistern could be filled in* 20 *min. by its supply pipe and emptied in* 35 *min. by its waste pipe. If the cistern be empty and both pipes be opened, how long would it take to fill it?*

The supply pipe fills $\frac{1}{20}$ of the cistern in 1 min.,
the waste pipe empties $\frac{1}{35}$ of the cistern in 1 min.;
hence, together they fill $(\frac{1}{20}-\frac{1}{35})=\frac{3}{140}$ of the cistern in 1 min.

And, as $\frac{3}{140}$ of the whole is filled in 1 min., the whole will be filled in 1 min. $\div \frac{3}{140} = 46\frac{2}{3}$ min.

EXAMPLES XC.

Written Exercises.

1. A can mow a field in 3 da., and B can mow the same field in 6 da.; in how many da. will they do it working together?

2. A bath could be filled by its cold water pipe in 15 min. and by its hot water pipe in 30 min.; in what time will it be filled when both are opened?

3. A can do a piece of work in 12 da., and B can do the same in 20 da. A works at it for 3 da. How long would it take B to finish it?

4. A can mow a field in 15 hr., and B can mow the same field in 25 hr. They work together for $7\frac{1}{2}$ hr., when A goes away. How long will it take B to finish the work?

5. Two men together can do in 20 days a piece of work which one of them alone could do in 30 days; how long would it take the other man to do the work alone?

6. When the hot and cold water pipes are both opened a bath is filled in 6 minutes; and when only the cold water is turned on, the bath is filled in 10 minutes. In how long would the bath be filled if the hot water pipe only were opened?

7. A and B could together finish a piece of work in 25 days. They work together for 15 days, and then A finished it by himself in 20 days. How long would it take them to do the whole, working separately?

8. A and B could together do a piece in $22\frac{1}{2}$ days. A worked at it alone for 10 days, and then B finished it alone in 60 days. How long would it take them separately to do the whole work?

9. A can do a piece of work in $2\frac{1}{2}$ days, B can do it in 3 days, and C can do it in $3\frac{3}{4}$ days; how long would it take them to do it, all working together?

10. A cistern is filled by one pipe in 48 minutes, by another in an hour, and by a third in half an hour; in what time would it be filled if all three pipes were open together?

11. A cistern can be filled by one pipe in 3 hours, by another in 3 hr. and 40 min., and it can be emptied by a third pipe in 2 hr. 20 min.; if it be empty, and they are all opened together, in what time will the cistern be filled?

12. C does half as much in a day as A and B can do together, and B does half as much again as A; if all three working together can mow 20 acres of barley in 16 days, how long would each, working by himself, take to mow 5 acres?

13. A can do a piece of work in 6 days, B in 8 days, and C in 12 days. B and C work together for 2 days, and then C is replaced by A. Find when the work will be finished.

14. A and B together can perform a piece of work in 24 hr., A and C in 30 hr., and B and C in 40 hr.; in what time would each be able to perform it when working separately?

RACES AND GAMES.

214. The following are examples of questions of this nature.

Ex. 1. *In a* 100 *yards race A can give B* 5 *yards start and just win; also, B can give C* 5 *yards start; how much could A give C?*

A runs 100 yards while B runs 95, and B runs 100 yards while C runs 95.

Hence, C's distance in any time $= \frac{95}{100}$ of B's $= \frac{95}{100} \times \frac{95}{100}$ of A's.

Hence, while A runs 100 yards, C will run $\frac{95}{100} \times \frac{95}{100}$ of $100 = 90\frac{1}{4}$ yards.

Thus, A can give $9\frac{3}{4}$ yards to C.

Ex. 2. *In a certain game A can give B* 1 *point in* 5, *B can give C* 1 *point in* 5, *and C can give D* 1 *point in* 8; *how many points in* 100 *can A give D?*

$$\begin{array}{llll} A:B & = 5:4 & = 25:20 & \\ \quad B:C & = \quad 5:4 & = \quad 20:16 & \\ \qquad C:D = & \qquad 8:7 = & \qquad 16:14. \end{array}$$

Hence, as in Ex. 3, Art. 210,

$$A:B:C:D :: 25:20:16:14$$
$$:: 100:80:64:56.$$

Thus, A can give (100 − 56 =) 44 points in 100 to D; *i.e.*, A can make 100 points while D makes 56 points.

EXAMPLES XCI.

Written Exercises.

1. A can give B 10 yards start in a race of 100 yards, and B can give C 10 yards start over the same distance. How many yards start can A give C?

2. A can give B 20 yards and C 51 yards start in a quarter of a mile race. How many yards could B give C in a quarter of a mile?

3. A can beat B by 5 yards in a 100 yards race, and B can beat C by 10 yards in a 200 yards race; by how much could A beat C in a 400 yards race, supposing that they always run at the same pace?

4. A wins a race of 100 yards, beating B by 19 yards and C by 10 yards; how many yards start ought C to give B in 200 yards that they may run a dead heat?

5. In a certain game A can give B 1 point in 10, B can give C 1 point in 6; how many can A give C in 100?

6. At a certain game A scores 100 points while B scores 85, and B scores 100 while C scores 80; how many will C score in the time that it takes A to score 500?

7. A can make 9 articles while B makes 14, and B can make 7 while C makes 6; how many can C make in the time that A makes 30?

8. In a certain game A can give B 1 point in 10, B can give C 1 point in 12, and C can give D 1 point in 15; how many can A give D in 1000?

9. A can give B 20 yards and can give C 41 yards start in a race of a quarter of a mile, and B can give C a start of 3 seconds over the same distance; how long does each take to run a quarter of a mile?

10. In a certain game A can give B 1 point in 5, B can give C 1 point in 8, and C can give D 3 points in 10; how many can A give D in 100?

215. The following examples are worth notice.

Ex. 1. *A starts at* 10 *o'clock to walk along a road at the rate of* 4 *miles an hour; B starts on a tricycle at* 45 *minutes past* 10 *and rides after A at the rate of* 9 *miles an hour. When will B overtake A?*

When B starts, A has already traveled $\frac{45}{60}$ of 4 miles; that is, 3 miles.

B gains on A at the rate of $(9 - 4 =)$ 5 miles an hour.

B will overtake A when he has gained 3 miles, which he will do in $(3 \div 5)$ hours $=$ 36 minutes.

Ex. 2. *At what time between* 4 *and* 5 *o'clock will the hands of the clock be together?*

At 4 o'clock the minute-hand is 20 minute-spaces behind the hour-hand. In one hour the minute-hand passes over 60 minute-spaces, and the hour-hand passes over 5 minute-spaces.

Thus, the minute-hand gains 55 minute-spaces in an hour.

Now, when the two hands are together, the minute-hand must have gained on the hour-hand 20 minute-spaces, and the time required for this $= \frac{20}{55}$ of an hour $= 21\frac{9}{11}$ minutes.

Thus, the time required is $21\frac{9}{11}$ minutes past 4.

Ex. 3. *A train traveling at the rate of* 45 *miles an hour is observed to completely pass a certain telegraph post in* 5 *seconds; it also completely passed in* 4 *seconds a second train which was traveling along a parallel line of rails in the opposite direction at the rate of* 30 *miles an hour. How long were the trains?*

The time the first train takes to completely pass a post is the time it takes to travel a distance equal to the length of the train; and, since the train goes at the rate of 45 miles an hour, it goes in 5 seconds a distance $= \frac{5}{3600}$ of 45 miles $=$ 110 yards. Thus, the first train is 110 yards long.

Again, in the time the trains take to completely pass one another the distance traveled by the two trains together must be the sum

of the lengths of the trains; and in 4 seconds the trains will together travel $\frac{4}{3600}$ of 45 miles + $\frac{4}{3600}$ of 30 miles = 146 yd. 2 ft.

Hence the length of the second train

= 146 yd. 2 ft. − 110 yd. = 36 yd. 2 ft.

Ex. 4. *Seven fowls are worth* 6 *ducks,* 7 *ducks are worth* 2 *geese,* 10 *geese are worth* 7 *turkeys, and a turkey is worth* 17*s.* 6*d.*; *how much is a fowl worth?*

One fowl is worth $\frac{6}{7}$ of the worth of a duck,

∴ " " $\frac{2}{7}$ of $\frac{6}{7}$ of the worth of a goose,

∴ " " $\frac{7}{10}$ of $\frac{2}{7}$ of $\frac{6}{7}$ of the worth of a turkey,

∴ " " $\frac{7}{10}$ of $\frac{2}{7}$ of $\frac{6}{7}$ of 17*s.* 6*d.*

$= \frac{7}{10} \times \frac{2}{7} \times \frac{6}{7} \times \frac{35}{2}$ shillings = 3*s.*

EXAMPLES XCII.

Written Exercises.

1. One boy runs at the rate of 100 yards in 15 seconds, and has a start of 40 yards in front of another boy who runs at the rate of 100 yards in 12 seconds; when will the first boy be overtaken?

2. One cyclist rides at the rate of 15 miles an hour and starts half-an-hour after another who rides along the same road at the rate of 12 miles an hour; when will the first rider be overtaken?

3. At what time between 5 and 6 o'clock will the hands of a clock be together?

4. At what time between 2 and 3 o'clock will the hands of a clock be at right angles?

5. A train traveling at the rate of 45 miles an hour is observed to completely pass a certain point in 9 seconds; find the length of the train.

6. A man on the platform of a station observed that a train passed him in 10 seconds, and passed completely through the station, which is 308 yards long, in 24 seconds; how long was the train, and how fast was it going?

7. A passenger train, moving at the rate of 45 miles an hour, overtook a mineral train twice as long as itself and which was going along a parallel line of rails in the same direction at the rate of 23 miles an hour; and the passenger train completely passed the mineral train in $22\frac{1}{2}$ seconds. How long was each train?

8. A person lights two candles, 12 and 10 inches long respectively, at 6 P.M. The former diminishes 5 inches in length in 4 hours, and the latter 1 inch in 2 hours. If kept alight, at what time will the former be two inches shorter than the latter?

9. If 3 pears are worth as much as 4 apples, 5 apples as much as 3 plums, 8 plums as much as 3 peaches, and if pears cost 36 ct. a dozen, what is the price of a peach?

10. Twelve fowls are worth as much as 11 ducks, 5 ducks are worth as much as 4 pheasants, 10 pheasants as much as 3 turkeys, and 7 turkeys as much as 10 geese; also a fowl and a pheasant are together worth 6*s.* 6*d.* Find the cost of a goose and a turkey together.

CHAPTER X.

PERCENTAGES.

PERCENTAGE A RATIO.

216. IN many cases the ratio of one number to another, or of one quantity to another of the same kind, is expressed by the *number of hundredths* the first is of the second, and this is called the **per cent** the first is of the second.

For example, 2 apples $= \frac{25}{100}$ of 8 apples, or 25 per cent of 8 apples.

This means that the ratio of 2 apples : 8 apples is .25. The first term is sometimes called the *percentage*, the second is called the *base*, and the quotient is, as formerly, the ratio.

Per cent is expressed by the sign %, or by writing the numerator as a decimal; thus, $\frac{25}{100} = 25\%$, or .25; and we write '5 is 25% of 20,' or '5 is .25 of 20.'

217. The following examples will show how to express any given quantity as a per cent of any other given quantity of the same kind.

Ex. 1. *Five dollars is what % of* \$40?

$$\$5 = \tfrac{5}{40} \text{ of } \$40 = \tfrac{1}{8} \text{ of } \$40 = \frac{12\frac{1}{2}}{100} \text{ of } \$40 = 12\tfrac{1}{2}\% \text{ of } \$40.$$

Ex. 2. *In a town whose population was* 243200 *there were* 15504 *children born in a year. Find the per cent the number of births was of the population.*

The ratio of births to population is 15504 : 243200;

$$15504 \div 243200 = .06\tfrac{3}{8}; \therefore \textit{Ans.} = 6\tfrac{3}{8}\%.$$

218. The following examples will show how to find a given per cent of a given quantity.

Ex. 1. *Find* $12\frac{1}{2}\%$ *of* \$18.

$$12\tfrac{1}{2}\% \text{ of } \$18 = .12\tfrac{1}{2} \text{ of } \$18 = \$2.25.$$

Ex. 2. *In a town whose population was* 243200 *the birth rate in a year was* $6\frac{3}{8}\%$ *of the population; how many children were born in the year.*

$$6\tfrac{3}{8}\% \text{ of } 243200 = .06\tfrac{3}{8} \text{ times } 243200 = 15504.$$

Ex. 3. *Of what is* 9 *ct.* $22\frac{1}{2}\%$?

Since 9 ct. is $22\frac{1}{2}\%$, 100%, or the whole, must be $\frac{100}{22\frac{1}{2}}$ of 9 ct. $= \frac{40}{9}$ of 9 ct. = 40 ct.

219. Frequently in finding percentage it is best to multiply by the common fraction which is equivalent to the per cent expressed decimally; thus,

$6\frac{1}{4}\%$ of $96 = \frac{1}{16}$ of 96; $12\frac{1}{2}\%$ of $432 = \frac{1}{8}$ of 432;
$16\frac{2}{3}\%$ of \$36.85 = $\frac{1}{6}$ of \$36.85 = \$6.14$\frac{1}{6}$. [Art. 134.]

EXAMPLES XCIII.

Oral Exercises.

What fractions are denoted by the following per cents?

1. 50.	**4.** 10.	**7.** $12\frac{1}{2}$.	**10.** $3\frac{1}{3}$.
2. 25.	**5.** 5.	**8.** $16\frac{2}{3}$.	**11.** $6\frac{1}{4}$.
3. 20.	**6.** $2\frac{1}{2}$.	**9.** $33\frac{1}{3}$.	**12.** $5\frac{5}{9}$.

What per cents are equivalent to the following fractions?

13. $\frac{1}{2}$.	**16.** $\frac{1}{3}$.	**19.** $\frac{3}{40}$.	**22.** $\frac{5}{16}$.
14. $\frac{3}{4}$.	**17.** $\frac{1}{10}$.	**20.** $\frac{11}{25}$.	**23.** $\frac{8}{30}$.
15. $\frac{2}{3}$.	**18.** $\frac{1}{20}$.	**21.** $\frac{4}{15}$.	**24.** $\frac{11}{12}$.

Written Exercises.

Find the per cent the first is of the second in

25. \$10, \$25; 45 ct., \$2.70; \$1.12$\frac{1}{2}$, \$3.

26. 7*s.* 6*d.*, £2. **27.** 7 lb., 1 cwt.

28. 1 hr. 12 min., 1 da.

29. 3 oz. 15 dwt., 1 lb. 10 dwt.

30. 3216^{dg}, 1.608^{kg}.

Find

31. 5% of £7. 10*s*.

32. 10% of $85.63.

33. $12\frac{1}{2}$% of $492.64.

34. The population of a certain town increased 50% in the 10 years from 1881 to 1891, and the population in 1891 was 34617; what was the population in 1881?

35. Find the % of error in the statement that 1 oz. Troy is equal to 1.1 oz. Avoir.

36. Fill the blanks in the following table by giving the per cents of the 1889 amounts to the nearest *tenth*.

Receipts from —	1890.	1889.	Increase.	
			Amount.	Per Cent.
	$	$	$	
Ordinary passengers . .	26983000	25678000	1305000	
Season-ticket holders . .	2316000	2196000	120000	
Excess baggage, mails, etc.	5029000	4757000	272000	
Total	34328000	32631000	1697000	

37. Fill the blanks in the following table by giving the per cents of the 1889 amounts to the nearest *tenth*.

Receipts from —	1890.	1889.	Increase.	
			Amount.	Per Cent.
	$	$	$	
Mineral traffic	17543000	17052000	491000	
General mdse traffic . .	23300000	22694000	606000	
Live stock	1377000	1340000	37000	
Total	42220000	41086000	1134000	

PROFIT AND LOSS.

220. When anything is sold for more than it cost, it is said to be sold *at a profit,* and when it is sold for less than it cost, it is said to be sold *at a loss.* Profit or loss is often expressed as a *percentage,* and this percentage is always to be reckoned on the **cost price.**

Thus, if goods which cost \$50 are sold for \$60, the percentage gain, or profit, is \$10, and the per cent gain is $10:50 = 20\%$ on the *original outlay.*

221. The following examples will show how to treat questions involving profit or loss.

Ex. 1. *A house was bought for* \$400 *and sold for* \$480; *what was the profit per cent?*

The total profit = \$480 − \$400 = \$80.

And the ratio of \$80 : \$400 = 20%.

Ex. 2. *An article cost* \$10.40 *and was sold at a loss of* 15%; *for what was it sold?*

Selling price = cost − 15% of cost;

∴ " " = 85% of cost = .85 of \$10.40 = \$8.84.

Ex. 3. *What was the cost of goods which were sold for* \$56, *at a gain of* 12%?

Selling price = cost + 12% of cost = 112% of cost;

∴ cost = $\frac{100}{112}$ of selling price = \$50.

Ex. 4. *By selling tea at* 50 *ct. a pound a grocer would gain* 5% *more than by selling it at* 48 *ct. a pound; what was the cost of the tea?*

50 ct. − 48 ct. is 5% of the cost; hence 2 ct. is 5% of the cost; ∴ 40 ct. = the cost.

Ex. 5. *A manufacturer sells at a profit of* 20% *to a wholesale dealer, who sells at a profit of* 15% *to a retail dealer, and the retail dealer sells for* \$2.76 *and makes a profit of* 25%. *Find the cost of manufacture.*

It cost the retail dealer $\frac{100}{125}$ of \$2.76;
" " wholesale dealer $\frac{100}{115}$ of $\frac{100}{125}$ of \$2.76;
" " manufacturer $\frac{100}{120}$ of $\frac{100}{115}$ of $\frac{100}{125}$ of \$2.76;

Thus required cost $= \frac{100}{120} \times \frac{100}{115} \times \frac{100}{125}$ of \$2.76 = \$1.60.

EXAMPLES XCIV.

Written Exercises.

What was the gain or loss % in the following cases?

1. Cost price \$20, selling price \$24.
2. Cost price \$2.00, selling price \$2.28.
3. Cost price 40 ct., selling price 44 ct.
4. Cost price \$3, selling price \$3.60.
5. Cost price \$140, selling price \$130.
6. Cost price \$1.20, selling price \$1.62.
7. Cost price 84 ct., selling price 98 ct.
8. Cost price \$7.80, selling price \$8.97.
9. Cost price \$74, selling price \$70.30.
10. Cost price \$15.20, selling price \$20.52.
11. Cost price \$12.40, selling price \$10.23.
12. Cost price \$147, selling price \$122.01.

13. If an article be bought for \$4.20 and sold for \$6.60, what is the gain %?

14. What was the cost price of tea which is sold for 80 ct. a pound and at a gain of 25%?

15. If a grocer buys 60 lb. of tea for \$21.00, at what price per lb. must he sell it so as to make 20% profit?

16. An article was sold for 56 ct., at a gain of 12%; what did it cost?

17. The profit on an article if sold for $3.00 is 25%; what would be the profit if it were sold for $2.88?

18. By selling a house for $759 a builder gained 10%; what would he have lost % if he had sold for $621?

19. If a profit of $22\frac{1}{2}$% is made by selling an article for $2.94, what would be the selling price if the profit were only 5%?

20. A person bought a carriage and sold it for $37.80 more than he gave for it, thereby clearing 7%; what did he give for it?

21. A house is sold for $4000, and 25% profit is made; how much % profit would be made by selling for $3360?

22. A tradesman by selling an article for $1.62 gains 35%; what would he have gained % if he had sold it for $1.98?

23. A man bought apples at the rate of 6 for 2 ct., and an equal number at the rate of 10 for 2 ct.; and he sold the whole at the rate of 5 for 2 ct. What profit % did he make?

24. If 5% more be gained by selling an article for 24 ct. than by selling it for 23 ct., what was the original price?

25. If 3% more be gained by selling a horse for $399.60 than by selling for $388.80, what must have been the original cost?

26. If a woman gains 12% by selling 5 herrings for 14 ct., what % would she gain by selling them at 6 for 18 ct.?

27. If a woman buys eggs at 20 ct. a dozen, how many ought she to sell for 18 ct. in order to gain 8%?

28. A man who had been paying $25.20 for 4 t. of coal changed his coal merchant and then got 5 t. for $20.16; how much did he save %?

29. A draper bought 240 yd. of silk. He sold $\frac{1}{4}$ at a gain of 25%, $\frac{1}{3}$ at a gain of 20%, and the remainder at a loss of 15%, and received $800 in all. What was the cost price per yd.?

30. A draper bought a piece of silk 35 yd. long; and, after cutting off 2 yd. which were damaged, he sold the remainder so as to clear 10% on his outlay. How much % was the selling price of a yd. higher than the cost price?

31. A manufacturer sold at a profit of 25% to a wholesale dealer, who sold at a profit of 12% to a retail dealer, and the retail dealer sold for $3.22 and made a profit of 15%; what was the cost of manufacture?

32. A quantity of wheat was sold in succession by three dealers, each of whom made a profit of 5%. The last of the three sold for $3087; how much did it cost the first?

33. A house was sold by the builder at a profit of 30%, and the purchaser sold it again at an advance of $117 in the price, and gained 20% on his outlay; how much did the house cost the builder?

TRADE DISCOUNT.

222. Merchants often sell goods at a certain price with a certain % discount; thus,

Macmillan & Co. may sell books at $1.60 per copy less 15%; this means that they sell for $1.60 – 15% of $1.60, or for $1.60 – $.24 = $1.36.

223. Sometimes after a given % discount is allowed, a second allowance of another % is made, and even a third allowance is made.

Ex. 1. *Goods sold for* \$2500 *with a discount of* 20 %, 5 %, and $1\frac{1}{2}$ % *bring what price?*

$$\$2500 - 20\% \text{ of } \$2500 = \$2000;$$
$$\$2000 - 5\% \text{ of } \$2000 = \$1900;$$
$$\$1900 - 1\tfrac{1}{2}\% \text{ of } \$1900 = \$1871.50 = \textit{Ans.}$$

Ex. 2. *Which is cheaper, to buy goods at a discount of* 30 % *and* 5 %, *or with* $33\frac{1}{3}$ % *off?*

The marked price less 30 % = 70 % of marked price; 70 % − 5 % of 70 % = $66\frac{1}{2}$ %. It is cheaper to buy at a discount of 30 % and 5 % than at a discount of $33\frac{1}{3}$ %.

EXAMPLES XCV.

Oral Exercises.

What is paid for goods marked

1. \$50 with a discount of 10% ?
2. \$50 with a discount of 10% and 10% ?
3. \$50 with a discount of 20% and 5% ?
4. \$600 with a discount of $33\frac{1}{3}$% ?
5. \$900 with a discount of $16\frac{2}{3}$% ?
6. \$1000 with a discount of 27% and 10% ?
7. \$1000 with a discount of 20%, 10%, and 1% ?

What is the marked price of goods sold for

8. \$90 after a discount of 25% ?
9. \$63 after a discount of 30% and 10% ?
10. \$49 after a discount of $12\frac{1}{2}$% and $12\frac{1}{2}$% ?
11. \$45 after a discount of $16\frac{2}{3}$% and 10% ?

Written Exercises.

12. Find what was received for goods marked \$1200 if a discount of $\frac{1}{4}$ and 15% is allowed.

13. For what % of the marking price are goods sold if an allowance of $\frac{1}{5}$, 10%, and $6\frac{1}{4}$% is made?

14. Goods are marked \$170 and sold for \$144.50; what % discount was allowed?

15. Goods marked \$16 were sold at $6\frac{1}{4}$% discount and 5% off for cash; what was the selling price?

16. Goods cost a merchant \$1600; he wishes to make a profit of 25% after making a discount of 20% and $16\frac{2}{3}$%; what was the marked price?

17. At what % above the cost must goods be listed that a merchant may allow a discount of 20% and realize a profit of 12%?

18. A merchant allows on \$2000 worth of goods (list price) a discount of 15%, 9%, and 5% for cash, then $\frac{1}{8}$% to clinch the bargain; how much cash did he receive and what profit did he make, his % of profit being 8?

COMMISSION AND BROKERAGE.

224. An agent employed to buy, or sell, goods, or to collect rents, is usually paid a percentage *on the price of the goods*, or *on the amount of rent. This* percentage is called **Commission.**

To insure against loss of life, or damage by fire, some persons pay money to an Insurance Company. In return for this money, the Company undertake to compensate the person insured for any loss caused by fire, or to pay a specified sum to relatives of the deceased. The money

paid to the Company is a percentage *on the value of the property insured*, or *on the specified sum*, and is called a **Premium**.

Ex. 1. *The total rental of an estate is* \$8474.40, *and the agent is paid a commission of* 5%; *how much is the commission?*

$$\$8474.40 \times .05 = \$423.72.$$

Ex. 2. *What is the annual premium for insurance on a building worth* \$7500 *at the rate of* 24 *ct. for* \$250?

$$\frac{.24}{250} \times \$7500 = \$7.20.$$

EXAMPLES XCVI.

Written Exercises.

1. After paying 5% to his agent, a man received \$1436.40; what was the agent's commission?

2. What is the amount of annual premium for the insurance of a building for \$8520 at $\frac{1}{10}$%?

3. A landlord allowed his tenants 20% reduction from their rents; what was the nominal rent of a tenant whose reduced rent was \$1800?

4. A commission merchant sells goods for \$2864 and sends to his principal \$2824.62 after deducting commission; what was the % commission?

5. A commission merchant is asked to purchase \$6800 worth of goods at $2\frac{1}{4}$% commission; how much money was paid by his principal?

6. A commission merchant received \$6953 with which to purchase goods after deducting $2\frac{1}{4}$% commission; what was paid for the goods?

7. An agent sold goods for \$5672; his bill for expenses was \$56.72, and his commission was $1\frac{1}{2}$%; what % of the selling price did the principal receive?

8. A man insured his life for \$5000 at an annual premium of $2\frac{1}{8}\%$; how much had he paid at the end of 13 years?

9. A cargo is insured for \$254500, its full value, at 2%; the ship is insured for \$120000 at $2\frac{3}{4}\%$; the owner of the cargo pays all insurance and sells his goods at the end of the voyage at an advance of 9% over total cost, allowing \$2000 for freight; what was the selling price?

10. The premium for insuring a building at $2\frac{1}{4}\%$ is \$1136.25; find the insurance.

11. A company insured a building and the goods it contained for \$117944, the goods being worth 15% of the value of the building. The merchant paid 2% premium on the building and $1\frac{1}{2}\%$ premium on the goods; what was the total premium?

12. A man sold through an agent some merchandise, paying the agent 5% commission. The agent invested the proceeds in two parts after taking out commissions of \$325 at 5%, and \$260 at 4%, respectively; what was the value of the merchandise?

13. A man had two houses, each costing \$5000; he insured one for \$4000 at $1\frac{1}{2}\%$, and the other for \$6000 at $1\frac{1}{2}\%$; find the difference between the loss on one and the gain on the other, both houses having been burned on the day after insurance.

Taxes and Duties.

225. Persons owning property or importing goods, pay to the government (for its support) a certain per cent of their property or of the *foreign value* of the goods imported.

The percentages paid on property are called **Taxes.**

The percentages paid on imported goods are called **Duties.**

Duties levied on articles regardless of their value are called **Specific Duties.**

Duties levied at a certain per cent on the foreign values of goods are called **Ad Valorem Duties.**

In some States voters pay annually a small fixed sum of money ($1.50 or $2) before they can vote. Such money is called a **Poll Tax**.

EXAMPLES XCVII.

Written Exercises.

1. The expenses of a certain town are $39512.32 annually; the tax is 16 mills on the dollar; what is the value of the town as fixed by the assessors? (The assessors' valuation is much smaller than the real valuation.)

2. The valuation of a certain town is $6495860, while the assessed valuation is 25 % of that; the polls number 1112, and the taxes are $16.25 on each thousand of assessed valuation; what are the expenses of the town?

3. The expenses of a city are $339000, and the assessed valuation is $16950000; what is the tax rate expressed as per cent? Expressed as dollars on a thousand?

4. What is the duty on 5000 bbl. of hydraulic cement at 8 ct. per bbl.?

5. What is the duty on 125 plates of polished unsilvered glass 24 × 30 in., at 8 ct. per sq. ft.?

6. What is the duty on 100 doz. penknives valued at 30 ct. per doz., at 25 % ad valorem?

7. What is the duty on 3 t. of No. 23 steel wire at 2 ct. per lb.?

8. A merchant imported 1550 yd. of tapestry carpet valued at 80 ct. a yd.; what was the duty at $42\frac{1}{2}\%$ ad valorem?

9. An invoice of 150 doz. linen collars valued at $1.30 per doz., calls for how much duty at 30 ct. and 30 %?

10. What does the government receive on an importation of 1000 gross of steel pens at 8 ct. per gross?

CHAPTER XI.

INTEREST.

PROMISSORY NOTES.

226. When one person borrows money from another person, he gives to the lender a written promise to repay the money and to pay also a percentage on the money at a given rate % per year. This percentage is called **Simple Interest**, or **Interest.**

The form of note given in the following pages is the form in use by the best business men in the United States. Students are strongly advised to adhere closely to the form while practising the making of notes.

The written promise is called a **Promissory Note.** For example:

$467^{00}. Nashville, Tenn., Jan. 1, 18 94.

Thirty days after date I promise to pay to the order of James Allen

Four hundred sixty-seven $\frac{00}{100}$ Dollars

at the German American Bank

Value Received, with interest.

James Ward.

No. 82. Due Jan. 31/Feb. 3, '94.

250\frac{64}{}$. Springfield, Mass., Jan. 1, 1895.

On demand I promise to pay to the order of Horace Belcher

Two hundred fifty and $\frac{64}{100}$ Dollars

at the Chapin National Bank

Value received, with interest at 5%.

No. 763. Due ---- J. O. Hammond.

227. In the case of a promissory note, it is to be noticed that the heading indicates the *names of the Town and State in which the note is written,* also the month, day, and year. At the left is written in figures the *sum of money* for which the note is given.

228.* The face of the note indicates the *names of the parties* (**Maker** and **Payee**) to the note, the words '*value received,*' the *sum of money* for which the note is given (written in full), and the *time* for which the note is to run. If the note is interest-bearing, it *must* have the words '*with interest*' written in the face.

229. Notes written as above may be transferred (sold) by the **Holder** to another person (who in turn becomes the holder), and are therefore called **Negotiable** notes. When the note contains the words '*or order,*' the holder must **Endorse** his name on the back of the note (thus becoming

* The Maker of a note is the person who signs the note. The Payee is the person to whom the note is made payable. The Holder of a note is the person who owns the note (the payee or some person to whom the payee has sold the note).

responsible for its payment) when he sells the note. When the note contains the words '*or bearer,*' no endorsement is legally necessary. If the note does not contain the words '*or order,*' or '*or bearer,*' it is not negotiable.

230. A note is payable at the *business office* of the *Maker* unless otherwise specified in the note. Nearly all notes specify the place of payment.

231. When the rate of interest is not written in the note, as in the first of the above notes, the law of the State *in which the note is to be paid* fixes the rate. If the parties interested wish a rate different from the rate in the State in which the note is to be paid, such rate must be specified in the body of the note, as in the second of the above notes, *but may not be more than the maximum allowed by such State.*

232. When the words '*with interest*' are omitted from a note, no interest is payable on that note except for the time it may over-run. [Art. 239.]

233. The first of the above notes is called a **Time** note; the second, a **Demand** note.

A *time note* is nominally due at the date indicated in the note, but **Matures** (becomes legally due) three days later. The three days are called **Days of Grace**. In some States no days of grace are allowed.

234. When a note matures on a *Sunday* or on a *legal holiday,* it is payable in some States on the business day next preceding, and in other States on the business day next succeeding, such Sunday or legal holiday.

235. The dates on which a note is nominally and legally due are indicated thus: Feb. 8/11, 1880.

If the time of payment is indicated in '*days after date*,' those days, together with three days of grace (if such be allowed by the State), are counted forward from (not including) the date of the note in finding the date of maturity; thus, the first of the above notes matures Jan. 31/Feb. 3, 1894.

236. If the time of payment is indicated in '*months after date*,' calendar months, together with three days of grace, are counted forward from the date of the note in finding the date of maturity; thus, the maturity of a note for 2 months, dated Jan. 31, 1892, was Mar. 31/Apr. 3, 1892; for 3 months, it was Apr. 30/May 3, 1892; for one month, it was Feb. 29/Mch. 3, 1892.

237. If payment of a note is not made on the day of maturity, the holder must engage a Notary Public to send to the endorser (or endorsers) a written notice of such fact. This notice is called a **Protest.** The protest must be sent on the day of maturity, otherwise the endorser cannot be held to the payment of the note.

TABLE OF RATES OF INTEREST.

238. The following table gives, for each of the States and Territories, the Legal Rate when no rate is mentioned in a note, the Maximum Rate allowed, the Time of payment when the day of maturity falls on a holiday (the day before by **B**, and the day after by **A**), and indicates by the letter **G** those States in which days of grace are legal.

Notes made on or after Jan. 1, '95, and payable in New York, bear no grace.

Notes made after July 4, '95, and payable in New Jersey, bear no grace.

State.	Rate.	Max.	Time.	Grace.	State.	Rate.	Max.	Time.	Grace.
Alabama . . .	8	8	A.	G.	Montana . . .	10	Any.	B.	G.
Arizona . . .	7	Any.	A.	G.	Nebraska . . .	7	10	A.	G.
Arkansas . . .	6	10	B.	G.	Nevada . . .	7	Any.	B.	G.
California . . .	7	Any.	A.		New Hampshire	6	6	B.	G.
Colorado . . .	8	Any.	B.	G.	New Jersey . .	6	6	A.	
Connecticut . .	6	6	B.	G.	New Mexico .	6	12	A.	G.
Delaware . . .	6	6	B.	G.	New York . .	6	6	A.	
Dist. of Columbia	6	10	A.	G.	No. Carolina .	6	8	B.	G.
Florida	8	10	B.	G.	No. Dakota . .	7	12	A.	G.
Georgia . . .	7	8	A.	G.	Ohio	6	8	B.	G.
Idaho	10	18	A.		Oklahoma . .	7	12	B.	G.
Illinois	5	7	B.	G.	Oregon . . .	8	10	A.	
Indiana	6	8	B.	G.	Pennsylvania .	6	6	A.	G.
Indian Territory	6	10	B.	G.	Rhode Island .	6	Any.	A.	G.
Iowa	6	8	B.	G.	So. Carolina . .	7	8	A.	G.
Kansas	6	10	B.	G.	So. Dakota. . .	7	12	A.	G.
Kentucky . . .	6	6	B.	G.	Tennessee . .	6	6	B.	G.
Louisiana . . .	5	8	A.	G.	Texas	6	10	B.	G.
Maine	6	Any.	B. or A.	G.	Utah	8	Any.	B. or A.	
Maryland . . .	6	6	B.	G.	Vermont . . .	6	6	A.	
Massachusetts .	6	Any.	A.	G.	Virginia . . .	6	6	B.	G.
Michigan . . .	6	8	A.	G.	Washington . .	8	Any.	B.	G.
Minnesota . .	7	10	A.	G.	W. Virginia . .	6	6	B.	G.
Mississippi . .	6	10	B.	G.	Wisconsin . .	6	10	A.	
Missouri . . .	6	8	A.	G.	Wyoming . .	12	Any.	A.	G.

EXAMPLES XCVIII.

1. Write a time note for $250.67 with interest at 16 %.

2. Write a time note for $76 with interest at 20 %.

3. Write a time note for $468.92 for 20 da. without interest.

4. Write a time note for $20 for 4 mo. with interest at 13 %.

5. Write a time note for $560, headed Cincinnati, Ohio, Jan. 13th, 1892, to mature in 63 da., with interest.

6. Write a demand note for $528 with interest.

7. Write a demand note for $460 with the maximum interest allowed by the State in which you live.

Find the date of maturity of each of the following indicated notes:

	DATE.	WHERE PAYABLE.	TIME.
8.	Dec. 18, 1895	New York	30 days
9.	Jan. 18, 1895	New York	60 days
10.	Jan. 27, 1896	New York	45 days
11.	July 31, 1895	New Jersey	60 days
12.	June 6, 1895	New York	3 months
13.	Jan. 30, 1895	New York	1 month
14.	Jan. 30, 1896	New York	1 month
15.	Mar. 31, 1895	New York	3 months
16.	Mar. 31, 1895	Conn.	1 month
17.	Mar. 30, 1896	Mass.	1 month
18.	Sept. 3, 1890	Nebraska	60 days
19.	Jan. 29, 1896	Cal.	30 days
20.	Jan. 29, 1896	Cal.	1 month

SIMPLE INTEREST.

239. Interest on the **Principal** (money borrowed) is called **Simple Interest,** and is computed at the given rate per cent (per year understood) for the time elapsing between the date and maturity of the note.

If a note is not interest-bearing and is not paid at maturity, interest is payable after maturity and until the note is paid. [Art. 232.]

240. The majority of notes are given for short periods of time — say 30, 60, or 90 days, or 1, 2, or 3 months. Now it is customary in interest computations, to regard one year as 360 days. Therefore, by a short operation, we may find the interest on any principal for any time and at any rate per cent.

Ex. 1.

350^{00}. Albany, N.Y., Jan. 16, 1895.
Eighteen days after date I promise to pay to
the order of Joel Putnam
Three hundred fifty $\frac{00}{100}$ Dollars
at the Park Bank
Value received, with interest.
No. 68. Due Feb. 3. O. C. Temple

$$\begin{array}{r} \$350 \\ .06 \\ \hline 360)\ \$21.00 \\ \hline .05\tfrac{5}{6} \\ 18 \\ \hline \$1.05 \end{array}$$

In this note, the rate is 6% and the time is 18 da.; the interest for 1 yr. will be .06 of the principal, — $21; the interest for one day is found by dividing $21 by 360, and the interest for 18 days is found by multiplying the quotient thus found by 18. Hence, .06 and 18 are multipliers, while 360 is a divisor.

This may be expressed as follows:

$$\frac{350 \times 6 \times 18}{100 \times 360}, \text{ which becomes}$$

$$\frac{.350 \times 18}{6} \text{ by cancellation.}$$

We observe that interest for *any number of days* may be found by dividing the principal by 1000, multiplying by the number of days, and dividing by 6.

The following is a better form for practical work:

$$\begin{array}{r|ll} \cancel{6} & .350. & \\ & \cancel{18} & 3 \\ \hline & \$1.050 & \end{array}$$

Here, cancelling 6 from the dividend and divisor, we have .350 to be multiplied by 3.

Ex. 1. *In the above note let the principal be* \$367.91, *the rate* 6%, *and the time* 21 *da.; find the interest.*

$$\begin{array}{l|ll} \cancel{2} & 3\cancel{6}\cancel{7}.\cancel{9}\cancel{1} & 18395+ \\ \cancel{6} & \quad\cancel{2}\cancel{1} & \quad 7 \\ \hline & & \$1.28765 \\ & & \$1.29 = \textit{Ans.} \end{array}$$

We must cancel the divisor completely; only 5 figures will be needed for the multiplicand; keep the multiplier as small as possible by cancellation.

Ex. 2.

\$650$\frac{00}{}$. Augusta, Me., Jan. 7, 1893.

Two months after date I promise to pay to the order of Henry Butler

Six hundred fifty $\frac{00}{100}$ Dollars

at the Granite National Bank

Value received, with interest.

Henry B. Reid.

No. 152 Due March 7/10, '93.

Find the interest.

$$\begin{array}{l|ll} \cancel{6} & \$\,\cancel{6}\cancel{5}\cancel{0}. & 325 \\ \cancel{2} & \quad\cancel{6}\cancel{3} & \ 21 \\ \hline & & \ 325 \\ & & 6\ 50 \\ \hline & & \$6.825 \\ & & \ 6.83 = \text{interest.} \end{array}$$

When the time is '*months after date*,' calendar months are counted in obtaining the date of maturity [Art. 236], and the interest is computed for the stated number of months *counting* 30 *da. as one month.*

This note payable in Maine has three days of grace; therefore the interest is computed for 63 da.

241. The value of a note at its *date of maturity* is called its **Maturity Value**, and consists of the sum for which the note is given plus the interest (if any). In finding *maturity value*, observe whether or not the note bears int. and '*days of grace*.'

EXAMPLES XCIX.

Written Examples.

What interest is due at *maturity* on each of the following indicated notes?

	DATE.	PRINCIPAL.	RATE.	TIME.
1.	Arkansas,	$763	6%	12 da.
2.	New Jersey,	$1467	"	15 da.
3.	Ohio,	$1626.75	"	18 da.
4.	Texas,	$6000	"	21 da.
5.	New York,	$5267.50	"	27 da.
6.	New York,	$2675	"	27 da.
7.	California,	$376	"	2 mo.
8.	Kentucky,	$498	"	1 mo.
9.	Connecticut,	$75000	"	108 da.
10.	New Hampshire,	$704.25	"	201 da.
11.	Illinois,	$84.75	"	361 da.
12.	Utah,	$846	"	51 da.

242. For rates other than 6%, find the interest at 6% and take such a part of that interest as the given rate is of 6%.

Ex. 1. $P = \$4673$, $R = 5\%$, *time* = 33 *da.*; *find interest.*

$$\begin{array}{r|rr} \cancel{6} & \cancel{4}{,}\cancel{6}\cancel{7}\cancel{3}. & 2{,}3365 \\ \cancel{2} & \cancel{3}\cancel{3} & 11 \\ \hline & & 6)25.7015 \\ & & 4.2835 \\ \hline & & \$21.42 \end{array} = Ans.$$

Here the interest at 6% is $25.7015, and $\frac{5}{6}$ of this is $21.42. We obtain this result rapidly by subtracting the interest at 1% from the interest at 6%.

Ex. 2. $P = \$26.48$, $R = 7\frac{1}{2}\%$, *time* = 90 *da.*; *find int.*

```
6|.026.48
 |  90  15
  4).39720
     .0993
   $ .50
```

Here the interest at $7\frac{1}{2}\% = \frac{5}{4}$ of the interest at 6%. The answer is obtained by adding to the interest at 6% one-fourth of the interest at 6%.*

In the *final* work do not waste time writing anything but the answer.

EXAMPLES C.

$700^{00}. Raleigh, N.C., Aug. 31, 1895.

Three months after date I promise to pay to the order of Thomas Greene

Seven hundred $\frac{00}{100}$ Dollars

at the Citizens' National Bank

Value received, with interest at 7 %.

C. J. Howard.

No. 16 Due Nov. 30/Dec 3, 1895.

1. Find the interest on the above note.

2. Find what would be the amount of the above note if the time were 90 da.

3. Find interest on a New York note for \$2670, dated Dec. 31, 1895, payable 47 da. after date, with interest.

---

* For interest at	8 %	add	$\frac{1}{3}$	of interest at	6 %	to	itself.
" "	$7\frac{1}{2}$%	"	$\frac{1}{4}$	"	"	"	"
" "	7 %	"	$\frac{1}{6}$	"	"	"	"
" "	$5\frac{1}{2}$%	subt.	$\frac{1}{12}$	"	"	from	"
" "	5 %	"	$\frac{1}{6}$	"	"	"	"
" "	$4\frac{1}{2}$%	"	$\frac{1}{4}$	"	"	"	"
" "	4 %	"	$\frac{1}{3}$	"	"	"	"
The "	3 %	=	$\frac{1}{2}$	"	"		

What interest is due at maturity on each of the following indicated notes?

	Date.	Principal.	Rate.	Time after Date.
4.	Saratoga, N.Y.,	$2670	Maximum.	47 da.
5.	Springfield, Mass.,	$4893	5%	60 da.
6.	Washington, D.C.,	$289.20	6%	90 da.
7.	Buffalo, N.Y.,	$48.93	$4\frac{1}{2}$%	3 mo.
8.	Utica, N.Y.,	$48.93	4%	3 mo.
9.	Baltimore, Md.,	$765	Maximum.	6 mo.
10.	Hartford, Conn.,	$4893	"	90 da.
11.	Denver, Col.,	$8695	12%	24 da.
12.	St. Louis, Mo.,	$463.50	9%	60 da.
13.	Chicago, Ill.,	$873	$6\frac{1}{2}$%	30 da.
14.	St. Paul, Minn.,	$487.20	$7\frac{1}{2}$%	2 mo.
15.	NewYork, N.Y.,	$286.37	$4\frac{3}{4}$%	3 mo.
16.	Boston, Mass.,	$499.99	4%	60 da.

17. Find the amount in each of the last three examples.

The above method for computing interest is in general use when the time is less than one year; but if the time is in yr., mo., and da., the 6% method is the more frequently used.

Six Per Cent Method.

243. A demand note for $268.50, dated Nov. 26, '87, was paid 4 yr. 7 mo. 18 da. after date; what was the interest at 6%?

At 6%

the interest on \$1 = \$.06 (six cents) for 1 yr.,
" " " \$1 = \$.005 (5 mills) for 1 mo.,
" " " \$1 = \$.000$\frac{1}{6}$ ($\frac{1}{6}$ of a mill) for 1 da.;

hence

" " " \$1 = \$.24 for 4 yr.,
" " " \$1 = \$.035 for 7 mo.,
" " " \$1 = \$.003 for 18 da.,
" " " \$1 = \$.278 for 4 yr. 7 mo. 18 da.

\$268.50	Having found the interest on \$1 for the given rate
.278	and time, we multiply this interest by the principal.
\$ 74.64	[Art. 47, Theorem I.]

Note. It is evident that the interest for 2 mo. at 6% may be computed by moving the decimal point 2 places to the left. Thus, the interest on \$784.70 for 2 mo. is \$7.85. Similarly, the interest for 6 da. is \$.78. Also the interest for 12 da. is \$1.57. The 6% method is sometimes used when the times are less than 1 yr.

244. There is great diversity in the methods of finding the time in a case like this. Some prominent banks and business houses in the United States use the method of counting the time in years and days, instead of the method just described.

Thus, the above note was paid July 14, '92; the 4 yr. were counted forward from Nov. 26, '87, the 7 mo. were counted forward as calendar months from Nov. 26, '91, and the 18 da. were counted forward from June 26, '92. (This is not compound addition.)

In obtaining the int. the years were reckoned as wholes, but the months and days were reckoned in the exact number of days found in *that* 7 mo. and 18 da. which began with Nov. 26, '91. Thus, the time was 4 yr. 231 da. – 4 in Nov., 31 in Dec., 31 in Jan., 29 in Feb., 31 in Mar., 30 in Apr., 31 in May, 30 in June, 14 in July.

The int. on \$1 = \$.24 for 4 yr.,
" " \$1 = \$.0385 for 231 da.,
" " \$1 = \$.2785 for 4 yr. 231 da.

\$268.50 × .2785 = \$74.77 = *Ans.*

EXAMPLES CI.

Written Exercises.

What was the amount *at maturity* of each of the following indicated notes?

Obtain answers by each method [Arts. 243, 244].

	DATE.		PRINCIPAL.	RATE.	TIME.
1.	Philadelphia,	July 31, '84	$7680.95	6 %	3 yr. 6 mo. 12 da.
2.	Richmond,	Aug. 6, '87	$683.42	"	4 yr. 7 mo.
3.	Boston,	Jan. 9, '88	$1492.88	"	4 yr. 11 mo. 18 da.
4.	Cleveland,	Oct. 1, '90	$2689.42	"	2 yr. 10 mo. 18 da.
5.	Jersey City,	Aug. 1, '95	$487.50	"	1 yr. 7 mo. 13 da.
6.	Providence,	Aug. 8, '79	$2000	"	8 yr. 5 mo.

245. For rates other than 6%, proceed as in Art. 242.

EXAMPLES CII.

A few demand notes are here indicated; find the interest on each of the first four, and the amount on each of the others.

Obtain answers by each method.

	DATE.	PRINCIPAL.	RATE.	PAID AFTER DATE.
1.	June 8, '81	$468.93	$4\frac{1}{2}$ %	5 yr. 9 mo. 18 da.
2.	Aug. 2, '87	$1680.50	4 "	3 yr. 6 mo. 27 da.
3.	Feb. 29, '88	$2500	$5\frac{1}{2}$ "	3 yr. 11 mo. 6 da.
4.	Mch. 9, '91	$155	$3\frac{1}{2}$ "	1 yr. 3 mo. 3 da.
5.	May 13, 89	$450.50	$6\frac{1}{2}$ "	3 yr. 2 mo. 7 da.
6.	Oct. 23, '85	$896.88	$4\frac{3}{4}$ "	4 yr. 11 mo. 29 da.
7.	Sept. 15, '82	$15875	$2\frac{1}{2}$ "	9 yr. 6 mo. 15 da.

ANNUAL INTEREST.

246. Some States allow interest to be collected on each *annual instalment of interest,* if such instalment is not paid when due.

Ex.

673\frac{00}{}$. Brooklyn, N.Y., April 5, 1895.

On demand I promise to pay to

the order of Henry Smith

Six hundred seventy-three $\frac{00}{100}$ Dollars

Value received, with interest annually at 5 %.

No. 17. Due.... Benjamin Jordan.

If this note be paid in 3 yr. 6 mo. after date, and no interest has been paid meanwhile, there will be paid the principal, the simple interest on the principal, and simple interest on each annual instalment of interest from the time it is due until the note is paid.

$$\begin{array}{rl}
\$673. & \\
.21 & \\
\hline
6)141.33 & \\
23.555 & \\
\hline
\$117.775 & = \text{interest for 3 yr. 6 mo.} \\
7.571 & = \text{interest on interest.} \\
673. & = \text{principal.} \\
\hline
\$798.35 & = \text{amount to be paid.}
\end{array}$$

The 1st instalment bears interest for	2 yr. 6 mo.
The 2d instalment bears interest for	1 yr. 6 mo.
The 3d instalment bears interest for	6 mo.
Interest on annual instalment is computed for	4 yr. 6 mo.

$$\begin{array}{rl}
\$33.65 & = \text{annual instalment at 5\%.} \\
.27 & \\
\hline
6)9.0855 & \\
1.5142 & \\
\hline
\$7.571 & = \text{interest on interest for 4 yr. 6 mo.}
\end{array}$$

EXAMPLES CIII.

Written Exercises.

1.

1863\frac{50}{}$. New York, N.Y., Jan. 18, 1895.

On demand I promise to pay to

James Hollis, or order,

Eighteen hundred sixty-three and $\frac{50}{100}$ Dollars

Value received, with interest annually at 4 %.

No. 417. Due.....   Henry Otis.

If this note be paid 5 yr. 8 mo. and 20 da. after date, no interest being paid meanwhile, how much will the holder receive?

2. Cast the interest on a note similar to the above, when $P = \$897.75$, $R = 5\frac{1}{2}\%$, and $T = 4$ yr. 9 mo. and 15 da., no interest being paid meanwhile.

3. How much does a man owe at the maturity of a note similar to the above, when $P = \$437.25$, $R = 4\%$, and $T = 7$ yr. 27 da., no interest having been paid?

COMMERCIAL DISCOUNT.

247. We have been considering, in the last few pages, cases in which money is borrowed from persons; we have learned that the interest is payable at the maturity of the note.

When money is borrowed from a bank, the interest (simple) is paid *on the day on which the money is borrowed.*

The simple interest which a bank takes in advance is called **Commercial Discount,** or **Bank Discount.**

The borrower does not receive the principal (as when borrowing from a person), but receives the principal minus the simple interest on the principal; this remainder is called the **Proceeds** of the note.

The following example will show the methods of calculation of *discounts* and *proceeds.*

Ex. 1.

$267⁰⁰. *Boston, Mass., Nov. 16, 1894.*

Thirty days after date I promise to pay to the order of ~~~~~~ *myself* ~~~~~~

~~~~ *Two hundred sixty-seven* ~~~~ $\frac{00}{100}$ *Dollars*

at ~~~~ *the First National Bank* ~~~~

Value received.

James Conklin.

No. 19. Due Dec. 16/19, '94.

248. When a person borrows money from a bank, he makes his note payable to himself, and at the bank which makes the loan. The note must be endorsed.

249. The discount is computed on the *maturity value* of the note; in the above note it is computed on $267, since the note is not interest-bearing. So, also, when a person sells a note to a bank (Ex. 2, following), the bank discounts its *maturity value.*

Find the discount and proceeds of the note in Art. 247.
~~~~

$$\begin{array}{r|rr} \cancel{6} & \cancel{2}\cancel{6}7. & .1335 \\ \cancel{2} & \cancel{3}\cancel{3} & 11 \\ \hline & & \$1.4685 \end{array}$$

$$\begin{array}{rl} \$267.00 & \\ 1.47 & = \text{discount.} \\ \hline \$265.53 & = \text{proceeds.} \end{array}$$

It will be observed that in case money is borrowed from a person the borrower has the use of a larger sum of money than when he borrows from a bank, yet he pays the same interest. The bank has the use of the discount while the borrower has the use of the proceeds.

At the maturity of a note given to a person the borrower pays principal and interest; at the maturity of a note given to a bank the borrower pays the principal only, having already paid the interest.

Ex. 2.

$700 00/100. New Haven, Conn., Jan. 31, 1895.

One month after date I promise to pay to the order of James Cushing

Seven hundred 00/100 Dollars

at the Fifth Avenue Bank, New York, N.Y.

Value received, with interest at 4½%.

No. 27. Due Feb. 28,'95. Henry Price.

Discounted Feb. 5, at 6%.

In this case the holder took a promissory note to some bank, and the bank discounted the note; *i.e.*, the teller gave the holder the proceeds of the note calculated on its maturity value. The *time for which a bank computes discount* is the exact number of days from the day of discount to the day of maturity, *although the time of the note may be written in months.* The holder must endorse the note. [Art. 229.]

$$\begin{array}{r|rr} \cancel{6} & 700. & \\ & \cancel{3}\cancel{0} & 5 \\ \hline & 4)3.50 & \\ & .875 & \\ \hline & \$2.63 & = \text{interest at } 4\tfrac{1}{2}\%. \\ & \$702.63 & = \text{maturity value.} \end{array}$$

$$\begin{array}{r|rr} \cancel{6} & \cancel{7}\cancel{0}\cancel{2}.\cancel{6}\cancel{3} & .1171 \\ & 23 & 23 \\ \hline & & \$2.69 = \text{discount.} \end{array}$$

$$\begin{array}{rl} \$702.63 & \\ 2.69 & \\ \hline \$699.94 & = \text{proceeds.} \end{array}$$

The note matures without grace, because payable in N.Y. The term of discount is the number of days from Feb. 5 to Feb. 28.

Ex. 3.

> $978 $\frac{25}{}$. *Burlington, Vt., Oct. 12, 1894.*
>
> *Five months after date I promise to pay to the order of* ~~~ *Frederick Hosmer* ~~~
>
> ~~~ *Nine hundred seventy-eight* ~~~ $\frac{25}{100}$ *Dollars*
>
> *at* ~~~ *the Howard National Bank* ~~~
>
> *Value received, with interest.*
>
> *No. 17. Due March 12, '95.* *Henry Thomas.*

Discounted Dec. 31, '94, at 5%.

$$
\begin{array}{r|l}
\cancel{6} & 978.25 \\
 & \underline{\cancel{150}\quad 25} \\
 & \$24.456 \\
 & \underline{\$978.25} \\
 & \$1002.71 = \text{maturity value.}
\end{array}
\qquad\qquad
\begin{array}{r|l}
\cancel{6} & 1002.71 \qquad .16712 \\
 & \underline{\qquad 71 \qquad\qquad 71} \\
 & 6)11.86552 \\
 & \underline{\quad 1.97758} \\
 & \$9.89 = \text{discount.}
\end{array}
$$

$$\$1002.71 - \$9.89 = \$992.82 = \text{proceeds.}$$

Ex. 4. *A merchant wishes* $750 *for immediate use for* 60 *days. What must be the principal of his note given to a New York bank?*

Here we have the proceeds and rate given, to find the principal. *Find the proceeds of* $1 *for* 60 *days;* this will be $ .99. Then,

$$\text{proceeds of } \$1 : \text{given proceeds} :: \$1 : \text{principal}.$$

$$\therefore \text{ Required principal} = \frac{\text{given proceeds}}{\text{proceeds of } \$1}$$

$$= \frac{\$750}{\$.99} = \$757.58.$$

EXAMPLES CIV.

1.

8700\frac{00}{}$. Nashville, Tenn., Dec. 27, 1894.

Ninety days after date I promise to pay to the order of myself

Eight thousand seven hundred $\frac{00}{100}$ Dollars

at the Bank of Commerce

Value received, with interest.

No. 127. Due ---- Henry Sims.

Find maturity, discount, and proceeds.

The State in which a note is to be paid determines the question of 'Days of Grace.'

2.

3000\frac{00}{}$. Denver, Col., Jan. 6, 1892.

Sixty days after date I promise to pay to the order of myself

Three thousand $\frac{00}{100}$ Dollars

at the North Denver Bank

Value received.

No. 10. Due ---- Herman Shippen.

Find maturity, discount, and proceeds.

3.

\$4530^{00}. *Santa Fé, N.M., Dec. 28, 1893.*
Three months after date I promise to pay to
the order of ~~~~~~ myself ~~~~~~
Four thousand five hundred thirty $\frac{00}{100}$ Dollars
at ~~~~~~ the First National Bank ~~~~~~
Value received.
No. 86. Due ~~~ Amos Tucker.

Find maturity, discount, and proceeds.

In all notes **Date of Maturity** and **Rate** are the first things to consider.

4.

\$460^{00}. *Morristown, N.J., Aug. 12, 1895.*
Two months after date I promise to pay to
the order of ~~~~~~ myself ~~~~~~
~~~~~~ Four hundred sixty ~~~~~~ $\frac{00}{100}$ Dollars
at ~~~~~~ the National Iron Bank ~~~~~~
Value received.
No. 19. Due ---- Wm. K. Robbins.

Find maturity, discount, and proceeds.

5.

\$685$\frac{00}{}$.	*Paw Paw, Mich., Aug. 31, 1894.*

Twenty days after date I promise to pay to the order of ~~~ *myself* ~~~
~~~ *Six hundred eighty-five* ~~~ $\frac{00}{100}$ *Dollars*
at ~~~ *the First National Bank* ~~~
Value received.
No. 11. Due ---- *F. F. Parks.*

Find maturity, discount, and proceeds.

6.

\$	*St. Albans, Vt., July 31, 1887.*

Seven months after date I promise to pay to the order of ~~~ *myself* ~~~
-- $\frac{}{100}$ *Dollars*
at ~~~ *the Welden National Bank* ~~~
Value received.
No. 22. Due --- *Leonard Jerome.*

Discounted at 6%.
Proceeds = \$6269.25; find maturity and discount.
~~~

7.

$	*New York, N.Y., Jan. 1, 1896.*
Two months after date I promise to pay to	
the order of myself	
$\frac{}{100}$ *Dollars*	
at the Corn Exchange Bank	
Value Received.	
No. 1. Due	*Chas. E. Brown.*

Discounted at 4%.
Proceeds = $248600; find maturity and discount.

8. What would have been the discount in 7, if the note had been discounted by a bank in Ohio at the same rate %?

9.

$	*Boston, Mass., Apr. 6, 1894.*
Fifteen days after date I promise to pay to	
the order of myself	
$\frac{}{100}$ *Dollars*	
at the Traders' National Bank	
Value received.	
No. 18. Due	*Leonard Mandel.*

Discounted at $4\frac{1}{2}$%.
Proceeds = $100000; find maturity and principal.

10.

$3560^{00}. New York, N.Y., Sept. 19, 1895.

Five months after date I promise to pay to the order of Frederick Prince
Three thousand five hundred sixty $\frac{00}{100}$ Dollars
at the Chemical National Bank
Value received, with interest.

No. 56. Due..... A. J. Alvord.

Discounted Dec. 31, at $4\frac{1}{2}\%$.
Find maturity and discount.

11. A note dated N.Y., July 7, 1891, payable in Ohio in 3 yr. after date, was discounted Jan. 16, 1892, at $4\frac{1}{2}\%$; the principal was $5000. Proceeds = ?

12.

$780^{00}. Richmond, Va., June 7, 1892.

Forty days after date I promise to pay to the order of Horatio Gates
Seven hundred eighty $\frac{00}{100}$ Dollars
at the Planters' National Bank
Value received, with interest.

No. 27. Due..... James Hamilton.

Discounted June 13, at $4\frac{1}{2}\%$.
Find maturity and proceeds.

13.

671\frac{27}{}$.	*New York, N.Y., Jan. 20, 1895.*

Ninety days after date we promise to pay to the order of ~~~~~ Samuel Graves ~~~~~
~~~~ Six hundred seventy-one ~~~~ $\frac{27}{100}$ Dollars
at the First National Bank, St. Augustine, Fla.
Value received, with interest.
No. 12. Due ----- Hirsch & Co.

Discounted Feb. 28, at 6 %.

Find maturity and proceeds.

14.

$	*Nashua, N.H., Dec. 31, 1894.*

Two months after date we promise to pay to the order of ~~~~~ Benjamin Graves ~~~~~
------------------------------------ $\frac{}{100}$ Dollars
at ~~~~~ the Security Trust Co. ~~~~~
Value received, with interest.
Samuel Johnson.
No. 102. Due ---- David Waite.

Discounted Feb. 3, at 5 %.

Find maturity, maturity value, and principal, when the proceeds = $790.

15.

$ ------ *Milwaukee, Wis., May 25, 1893.*

Four months after date I promise to pay to the order of Chas. P. Bishop

------------------------------------ $\frac{}{100}$ *Dollars*

at the German American Bank

Value received, with interest.

No. 113. Due ---- Horace White.

Discounted July 1, '93, at $5\frac{1}{2}\%$.

Proceeds = \$8000; principal = ?

Ans. $\begin{cases} \$7920.84 = \text{face.} \\ \$8110.28 = \text{maturity value.} \end{cases}$

EXACT INTEREST.

250. Thus far interest has been computed on the basis of a year of 360 da. Such interest is evidently $\frac{73}{72}$ of the interest computed on the basis of a year of 365 days.

The interest computed on the basis of a year of 365 da. is called **Exact Interest,** and is computed for only fractions of a year.

Exact interest is computed in interest transactions with General Governments and in many interest transactions of ordinary business.

Ex. 1. *Find exact interest at* $4\frac{1}{2}\%$ *on a note for* \$892, *dated Feb.* 16, '93, *and maturing Apr.* 2/5, '93.

$\cancel{6}\,|\,892.$
$\ \ \ \cancel{48}\quad 8$
$4)7.136$
1.784
$73)5.352$
$.073$
$\$5.28$

Here we find the interest at $4\frac{1}{2}\%$ for the exact number of days and on the 360 days basis, and subtract from this interest $\frac{1}{73}$ of itself to obtain the answer on the 365 days basis.

Ex. 2. *Find the exact interest at* 4 % *on a note for* \$781.20, *dated June* 5, '89, *and maturing Oct.* 4/7, '92.

\$781.20	
.12	
\$ 93.74	Interest for 3 yr.
9.00	Exact interest for 124 da.
\$102.74	= *Ans.*

Here simple interest is computed for 3 yr., and the exact interest for 124 da. (leap yr.) is added.

EXAMPLES CV.

Written Exercises.

What is the amount at maturity of each of the following indicated notes, exact interest?

		Date.	Face.	Rate.	Maturity.
1.	Texas,	Jan. 4, '93,	\$890	$4\frac{1}{2}\%$	Apr. 4/7, '93.
2.	N. Y.,	Jan. 8, '96,	\$400	4%	Apr. 7, '96.
3.	Mass.,	June 11, '91,	\$250	6%	Mch. 10/13, '92.
4.	Oregon,	Apr. 19, '93,	\$1250	Legal	July 18/21, '93.
5.	Iowa,	Aug. 6, '94,	\$46849	Legal	Nov. 4/7, '94.
6.	Vt.,	Feb. 4, '95,	\$2685	5%	May 5, '95.

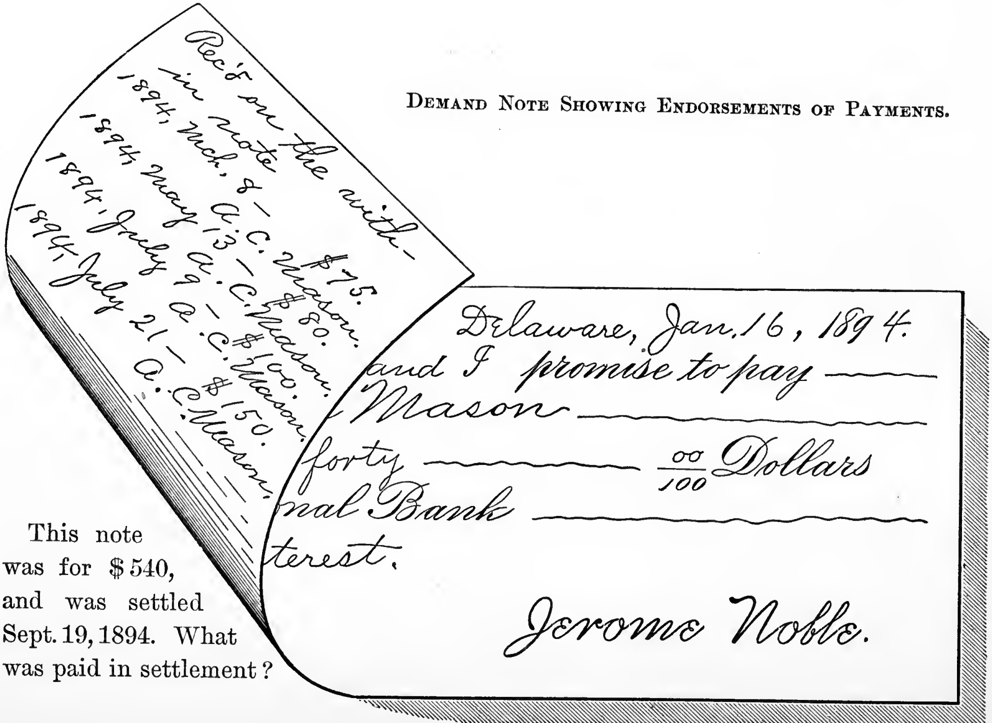
Partial Payments.

251. It often occurs that part of a note is paid at one time, another part at another time, and so on, until all the note is paid. Such payments are called **Partial Payments.**

In case of interest-bearing notes, it becomes necessary to compute *simple interest* on the different principals which appear during the life of the note.

The sums of money paid and the times of payment are **Endorsed** on the back of the note.

DEMAND NOTE SHOWING ENDORSEMENTS OF PAYMENTS.



This note was for $540, and was settled Sept. 19, 1894. What was paid in settlement?

Dates found on the note.	Times between successive dates.
Jan. 16, '94.	
Mch. 8, '94.	......51 da.
May 13, '94.	......66 "
July 9, '94.	......57 "
July 21, '94.	......12 "
Sept. 19, '94.	......60 "

When a note is wholly within a year, the exact number of days between dates is found, and the days' method is used in computation.

\$540.	1st prin.
4.59	int. for 51 da.
\$544.59	am't of 1st prin.
75.	1st payment.
\$469.59	2d prin.
5.17	int. for 66 da.
\$474.76	am't of 2d prin.
80.	2d payment.
\$394.76	3d prin.
3.75	int for 57 da.
\$398.51	am't of 3d prin.
100.	3d payment.
\$298.51	4th prin.
.60	int. for 12 da.
\$299.11	am't of 4th prin.
150.	4th payment.
\$149.11	5th prin.
1.49	int. for 60 da.
\$151.60	am't paid Sept. 19.

It will be observed that the amount of the principal is found for the time elapsing between the date of the note and the date of the first payment. The first payment is subtracted, and the remainder is used as a new principal. And so on to the end.

Ex. 2.

5000\frac{00}{}$. Annapolis, Md., June 19, 1888.

On demand I promise to pay to the order of Henry Johnson Five thousand $\frac{00}{100}$ Dollars at the Farmers' National Bank Value received, with interest at 4%.

Sam'l Babbitt.

This note carried the following endorsements:

Dec. 1, '88, $150; Apr. 7, '89, $250; Oct. 25, '90, $275; Mch. 1, '92, $1000; Mch. 1, '93, $2000.

Find the balance which was paid on Sept. 19, '94.

Here we find the times in years, months, and days.

Dates found on the note.			Times between successive dates. yr.	mo.	da.	Interest on $1 at 6% for the times.	
'88	6	19		5	12	$.027	
'88	12	1		4	6	$.021	
'89	4	7	1	6	18	$.093	= $.174
'90	10	25	1	4	6	$.081	
'92	3	1	1			$.06	
'93	3	1	1	6	18	$.093	
'94	9	19					

$5000	1st principal.
90	int. for 5 mo. 12 d., at 4%.
$5090	am't of 1st prin.
150	1st payment.
$4940.	2d prin.
69.16	int. for 4 mo. 6 d., at 4%.
$5009.16	am't of 2d prin.
250.	2d payment.
$4759.16	3d prin.
552.06	int. for 2 yr. 10 mo. 24 d., at 4%.
$5311.22	am't of 3d prin.
1275.	3d and 4th payments.
$4036.22	4th prin.
161.45	int. for 1 yr., at 4%.
$4197.67	am't of 4th prin.
2000.	5th payment.
$2197.67	5th prin.
136.26	int. for 1 yr. 6 mo. 18 da., at 4%.
$2333.93	am't paid Sept. 19, '94.

In case any payment is less than the interest due at the time of such payment (as in the 3d payment of this note) a portion of the interest would become a part of the new principal and would draw interest, if we should proceed as with the 1st and 2d payments. Here compound interest is forbidden by law, and we must find the interest on the same principal until the time when the sum of the payments equals or exceeds the interest.

The United States Rule.*

252. *Compute the amount of the principal to the time when a payment, or the sum of two or more payments, equals or exceeds the interest due.*

Subtract from this amount the payment, or the sum of the payments, and proceed with the remainder as a new principal. And so on to the time of settlement.

* Vermont, New Hampshire, and Connecticut have methods of their own for computation in partial payments, but it is not advisable to consider those methods in our present study.

EXAMPLES CVI.

1. A Kentucky note for \$3500, with interest, dated Mch. 1, '90, had the following endorsements:

Apr. 6, '90, \$500.	May 15, '90, \$800.
" 30, '90, \$300.	July 11, '90, \$600.

What was paid in settlement on Aug. 22, '90?

2. An Arizona note for \$8600, with interest, dated July 1, '87, had the following endorsements:

Oct. 2, '87, \$150.	Feb. 21, '88, \$4000.
Nov. 7, '87, \$1500.	

What was due May 4, 1888?

3. A Louisiana note for \$876, with interest, dated Feb. 6, '86, was endorsed as follows:

Apr. 11, '86, \$50.	June 2, '87, \$300.
Dec. 1, '86, \$150.	July 5, '87, \$75.

What was paid in settlement on Jan. 1, '88?

4. A Massachusetts note for \$3000, with interest at $4\frac{1}{2}\%$, dated Jan. 1, '91, was endorsed as follows:

Mch. 7, '91, \$175.	Sept. 20, '93, \$800.
May 9, '91, \$300.	Nov. 30, '94, \$80.
Aug. 17, '93, \$400.	

What was paid in settlement on Dec. 5, '94?

5. An Indiana note for \$2500, dated Jan. 6, '94, was endorsed as follows:

Feb. 7, '94, \$250.	Oct. 6, '94, \$500.
Apr. 20, '94, \$180.	Feb. 7, '95, \$350.
July 7, '94, \$75.	

What was paid in settlement on Feb. 20, '95?

Ans. \$1145.

CHAPTER XII.

EXCHANGE.

DRAFTS.

253. Suppose that Wilson & Co. of Baltimore buy of Morton & Co. of St. Paul $2500 worth of goods on 60 da. credit. When the bill is due, Morton & Co. may make a formal request for its payment. Such a request is called a **Draft**; *Morton & Co. are said to draw on Wilson & Co.* For example:

> *$2500⁰⁰.* *St. Paul, Minn., July 25, 1894.*
> *At sight* *Pay to the*
> *Order of* *Ourselves*
> *Twenty-five hundred* $\frac{00}{100}$ *Dollars*
> WITH EXCHANGE.
> *Value received and charge the same to account of*
> *To Wilson & Co.,*
> *No. 29 Baltimore, Md.* *Morton & Co.*

The draft is sent to Wilson & Co. through a St. Paul bank which transmits it to a Baltimore bank. The latter presents the draft to Wilson & Co. for payment, and the cash is sent to Morton & Co. through the St. Paul bank.

The banks charge a small fee for their services, and the words 'with exchange' in the draft signify that the debtor must pay the fee.

The above draft is called a **Sight Draft.**

A *sight draft* is payable on presentation (most States not allowing grace on sight drafts), and, from its nature, is not subject to discount.

254. Instead of waiting for the expiration of the 60 da. and then drawing 'at sight,' Morton & Co. might make a **Time Draft,** payable after date. For example:

$2500^{00}. St. Paul, Minn., July 25, 1894.

Forty-five days after date Pay to the

Order of Ourselves

Twenty-five hundred $\frac{00}{100}$ Dollars

Value received and charge the same to account of

To Wilson & Co., Morton & Co.

No. 916. Baltimore, Md. Sept. 8/11, '94.

If Wilson & Co. accede to the request, they make a formal acceptance of the draft by writing across its face the word '**accepted,**' together with their signature. Their acceptance is equivalent to their making a *promissory note,* and the draft is regarded as such by all concerned. After acceptance, the draft is returned (through the banks) to Morton & Co., who now have a written promise from Wilson & Co., whereas before they had only a verbal promise. Morton & Co. now have the draft discounted, *exactly* as if it were a promissory note, and thus obtain the cash needed.

255. In case the *time draft* is made payable '*after sight*' instead of '*after date,*' Wilson & Co. affix to their acceptance the *date* of acceptance so that maturity may be found.

The payee is the owner of the draft. [See also Art. 228, Note.] For example :

$2500^{00}. *St. Paul, Minn., July 25, 1894.*

Forty-five days after sight *Pay to the*

Order of *Ourselves*

Twenty-five hundred $\frac{00}{100}$ *Dollars*

Accepted, July 27 '94. Wilson & Co.

Value received and charge the same to account of

To Wilson & Co., } *Morton & Co.*

No. 916. Baltimore, Md. } *Sept. 10/13, '94.*

Ex. *This draft was discounted at 6% on July 29th; find maturity and proceeds.*

$$\begin{array}{r|ll} \cancel{3} & \cancel{2},\cancel{5}\cancel{0}\cancel{0}. & .8333 \\ \cancel{6} & \cancel{4}\cancel{6} & 23 \\ \hline \end{array}$$

$\$19.17$ = discount.
$\$2480.83$ = proceeds.

From day of discount to maturity was 46 da.

Time drafts are rarely used, while *sight drafts* are very common.

EXAMPLES CVII.

1. E. A. Winslow of Brattleboro, Vt., drew on F. B. Crane of St. Louis, Mo., for the payment of a $650 debt contracted Apr. 13, '92, and due in 90 da. The draft was dated May 13, '92, and made payable 'after date.'

Write the draft, indicating acceptance, and write its date of maturity in the lower right-hand corner.

2. Rewrite the draft, making it payable 'after sight' and find its maturity, it having been accepted on May 16, '92.

3. Winslow had the first draft discounted May 20; find the proceeds.

4. What would have been the proceeds of the second draft, had it been discounted May 18, '92?

5. On Jan. 1, '92, S. B. Titus of Austin, Texas, drew on Ward & Co. of Macon, Ga., for the payment of a $1765 debt, contracted Dec. 7, '91, and due in 90 da.

Write the draft, payable 'after sight,' indicate acceptance on Jan. 3, '92, and write its date of maturity.

6. Draft in Ex. 5 was discounted Jan. 6, '92; proceeds = ?

256. It is evident that all the drafts thus far shown have been requests made by a creditor to his debtor. Now drafts may be used for *paying* debts as well as for *collecting* debts. In this case the debtor (through his bank) makes a draft on some bank in the city where his creditor lives and payable to such creditor.

DOMESTIC EXCHANGE.

257. The main object of drafts is the payment of debts without sending the actual money, thus avoiding expense, and risk of loss.

The draft method of making payments between cities in the same country is called **Domestic Exchange.**

FOREIGN EXCHANGE.

258. The draft method of making payments between cities in different countries is called **Foreign Exchange.**

259. Foreign drafts are made more extended in form than domestic drafts, and are called **Bills of Exchange.** A *Bill of Exchange* consists of a set of *two* bills, both alike,

except that they are numbered. These two bills are sent by different steamers, and as soon as one of the bills has been paid the other becomes void.

260. The drawing of Bills of Exchange is done by brokers, and no commission is charged for transacting the business.

261. The actual amount paid for Bills of Exchange, for example paid in New York for bills on London, varies from time to time; the current price paid for Bills, called the '**Rate of Exchange**,' cannot, however, ordinarily be much above or below par; for if it would cost more to discharge a debt by means of a bill than by the actual transmission of bullion, the latter method would naturally be adopted.

It should be noticed that even if all countries had exactly the same coinage, there would still be fluctuations in the rate of exchange between two countries, as the balance of indebtedness between those two countries varied.

262. The following table gives the value of some foreign coins in terms of U. S. Money as proclaimed by the Secretary of the Treasury on Jan. 1, '95:

Austria	1 Crown	=	$ .20, 3
Belgium	1 Franc	=	.19, 3
Brazil	1 Milreis	=	.54, 6
Chili	1 Peso	=	.91, 2
China	1 Tael { Shanghai	=	.67, 3
	Haikwan	=	.74, 9
Cuba	1 Peso	=	.92, 6
France	1 Franc	=	.19, 3
Germany	1 Mark	=	.23, 8
Great Britain	1 Pound Sterling	=	4.86, 6½
Holland	1 Guilder	=	.40, 2
Italy	1 Lira	=	.19, 3

Japan	1 Yen (gold)	=	.99, 7
Mexico	1 Dollar (gold)	=	.98, 3
Norway	1 Crown	=	.26, 8
Russia	1 Rouble (gold)	=	.77, 2
Spain	1 Peseta	=	.19, 3
Sweden	1 Crown	=	.26, 8
Switzerland	1 Franc	=	.19, 3

These values are subject to change.

263. Exchange on Great Britain is quoted at the *value of one pound sterling* (£1) *in dollars;* exchange on France is quoted at the *number of francs to the dollar;* exchange on Germany is quoted at the *value of four reichsmarks.*

The following is copied from a daily journal:

The foreign exchange market was steady, but very quiet in tone. Posted rates were unchanged at \$4.88$\frac{1}{2}$ for sixty-day bills and \$4.90 for demand. Actual sales were \$4.87$\frac{3}{4}$ @ \$4.88 for sixty-day bills, \$4.89$\frac{1}{4}$ for demand, \$4.89$\frac{1}{2}$ for cables, and \$4.87 @ \$4.87$\frac{1}{4}$ for commercial.

In Continental, francs 5.17$\frac{1}{2}$ for long and 5.16$\frac{1}{4}$ for short; reichsmarks 95$\frac{1}{2}$ and 95$\frac{3}{4}$; guilders at 40$\frac{1}{2}$ and 40$\frac{5}{8}$.

The following example shows the form of a Bill of Exchange and how to find its cost.

£1200.00. New York, N.Y., Oct. 29, 1894.

At sight of this First of Exchange, second of the same date and tenor, unpaid Pay to the Order of Samuel Littlejohn Twelve hundred pounds sterling Value received and charge the same to

To J. S. Morgan & Co., London, England } Drexel, Morgan & Co.

On Oct. 29th sight drafts on London were quoted at $4.88½.

$$\begin{array}{r} £1200 \\ \underline{4.88\tfrac{1}{2}} \\ \$5862.00 \end{array} = \text{cost of exchange.}$$

Ex. 2. *How large a sight draft on London can be purchased for* $3890, *exchange at* 4.86¼?

$$4.86\tfrac{1}{4}\underline{)3890.}$$
$$£800. = \textit{Ans.}$$

EXAMPLES CVIII.

Find the cost in New York of a Bill of Exchange for

1. £500 on London at 4.86½.

2. £1750 on Glasgow at 4.85.

3. 50000 francs on Paris at 5.18¾.

4. 1250 marks on Berlin at 95¼.

5. 2000 milreis on Rio Janeiro at 54.9 [cents per milreis].

6. 3000 crowns on Vienna at par.

7. Calculate the cost at market prices (as found in some daily journal) of
   - *a.* £650.
   - *b.* 2400 francs.
   - *c.* 2000 marks.

8. What will be the face of a N. Y. draft on Bremen costing $297.96, exchange being at 95¼? (Omit decimals of the answer.)

9. How large a draft on London can be purchased for $8554.14, exchange being quoted at 4.88¼?

10. How large a draft on Paris can be purchased for $1920, exchange being quoted at 5.18¾?

CHAPTER XIII.

STOCKS AND BONDS.

STOCKS.

264. There are many business undertakings, such as railways, banks, gas works, etc., which are on so large a scale that many persons must combine to provide the money necessary to carry on the business. This is generally done by dividing up the whole sum required into '**Shares**' of definite amount, say of $10, or $50, or $100 each.

The whole body of partners is called a **Company**, and the individual partners are called **Stockholders.**

The total amount of money raised to carry on the business of the company is called its **Capital.**

The affairs of a company are managed by a small number of elected stockholders called **Directors.**

The profits made by the company are called **Dividends,** and are periodically divided among the stockholders; the dividend is declared as a percentage on the capital.

265. A stockholder in a company cannot demand the return of the money he paid for his shares; he can, however, *sell* the shares.

If the dividends of the company are high, and are likely to continue to be high, the shares will sell for more than they originally cost; if, however, the com-

pany is not prosperous, the shares would have to be sold for less than they originally cost.

Thus, the stockholders in a company are continually changing, and different stockholders may have bought their shares at very different prices.

266. The most important point to notice is that the amount of dividend paid to a stockholder *does not depend on the price at which his shares were bought, but simply on their nominal value.*

Thus, two men who had the same number of $100 shares in a company would be entitled to the same amount of dividend, although one may have bought, for example, $100 shares for $180 and the other for $50 each.

267. Shares are said to be above or below **'par'** according as they are sold for more or for less than their nominal value. The nominal value is $100 per share, unless otherwise stated.

Thus, if $100 shares sell for $110 each, since $110 is $\frac{110}{100}$ of $100, the shares are 10 *per cent above par.*

When the price of shares is more than their nominal value they are said to be **'at a premium,'** and when the price is less than their nominal value the shares are **'at a discount.'**

268. The following are examples of the different questions which may have to be considered.

Ex. 1. *$100 shares in a gas company sell for $240 each; how much will 70 shares cost?*

Each $100 share costs $240 cash;

$\therefore$ 70 shares cost $240 × 70 = $16800.

Ex. 2. *A man bought $100 shares in a gas company for $16800, giving $240 for each $100 share; how many shares did he buy?*

Since each share cost $240,

the number of shares = $16800 ÷ $240 = 70.

Ex. 3. *A gas company pays a dividend of* 8% *per annum; how much does a man receive who holds* 70 $100 *shares?*

His share of the capital is $100 × 70 = $7000, and he receives 8% on this, or $560.

Ex. 4. *A man invests money in the stock of a company, each* $100 *share costing* $240; *what* % *does he receive on his investment when the company pays an* 8% *dividend?*

He receives $8 on each share, and having paid $240 for a share, he receives $8 on each $240 invested; $\frac{8}{240} = 3\frac{1}{3}\%$.

269. Sometimes a company does not need its full capital to carry on its business; and in that case only a certain fraction of the nominal amount of the shares is 'paid up'; the stockholders are, however, bound to pay the rest if it should become necessary. When a dividend is declared at so much per cent, this percentage is paid only on the amount paid up on the shares, and not on their full nominal value.

Ex. *What income will be obtained by investing* £1008 *in the purchase of* £20 *bank shares, on each of which* £5 *is paid up, at* £24 *each share, the bank paying a dividend of* 18 *per cent?*

Since £24 buys one share, £1008 will buy £1008 ÷ £24 = 42 shares. These 42 shares, on each of which £5 is paid, make up a capital of £5 × 42 = £210. On this capital of £210 a dividend of 18% is paid; hence, income required = £210 × $\frac{18}{100}$ = £37. 16*s*.

EXAMPLES CIX.

Written Exercises.

1. If $10 shares sell for $3.50, how many shares can be bought for $9271.50? What is the nominal value of shares purchased?

2. Mining shares of $10 each are sold at $2.50 discount; what is the price of 80 shares?

3. The shares of a certain company are sold at 10% above par; how much must be paid for 1060 $50 shares?

4. A company pays a dividend of 8%; how much does A receive if he holds 50 $50 shares?

5. A man holds 350 shares of $50 each, and the company pays 7% dividend; how much does he receive?

6. A man sells 63 $100 shares for $180 each, and buys with the proceeds $50 shares at $35 each; how many shares does he buy?

7. What is the difference between a $100 stock and $100 worth of stock?

8. A man sold 75 $50 shares for $65 each, and invested the money in $100 shares at $125 each; how many shares did he buy?

9. What income would be obtained by investing $3850 in the purchase of $100 shares in a company at $175 each, the company paying a dividend of 6% per annum?

10. $100 shares in a certain bank sell at $350, and the bank pays a semi-annual dividend of 7%; what annual income would be obtained by investing $9450?

11. A company pays a dividend of $4\frac{1}{2}$%, and its $100 shares sell for 50% above par; what per cent does an investor receive?

12. A man buys $50 shares at $62.50, and the company pays a 5% dividend; what percentage does he receive, and what % on his investment?

13. A man sells fifty shares of $100 gas stock, paying 8% dividend, at $180; he invests the proceeds in $50 railway stock at $35; find the change in his income, the railway company paying a dividend of $3\frac{1}{2}$%.

14. A man buys $100 stock in a company which pays an 8% dividend, and he buys at such a price as to receive 3% on his investment; what does he pay per share?

15. A bank pays a 9% dividend, and its $600 shares, of which $200 is paid up, sell for $750; what % does an investor receive on his money?

The price of Stock is given at so much *per cent;* thus, stock is said to be at 115, when $100 stock costs $115, and so in proportion for other amounts.

16. How much will $500 stock at 75 sell for?
How much will $150 stock at 120 sell for?
How much will $60 stock at 128 sell for?
How much will $1200 stock at 97 sell for?

17. What income will be obtained from $500 stock when the dividend is 4%?

18. What income will be obtained by investing $110175 in a stock which pays $3\frac{1}{2}$%, and can be bought at 113?

19. What income will be obtained by investing $70380 in a $3\frac{1}{2}$% stock at $97\frac{3}{4}$?

20. What % will a man get on his money if he invests in a 4% stock at 125?

21. A man receives $660 a year by investing $21450 in 4% railway stock; what was the nominal value of the stock?

22. An income of $506.25 per year is derived by investing $15300 in a $4\frac{1}{2}$% stock; what was the price of the stock per share?

23. Stock was purchased at $97\frac{1}{2}$ and sold at $103\frac{1}{4}$, and the profit was $661.25; how much stock was purchased and what was the total cost?

24. In which will a man receive the greater % on his investment; in a 3% stock at 95 or in a 4% at 127?

25. What will be the difference in income between a 4% stock at 129 and a $4\frac{1}{2}$% at 145?

BONDS.

270. Governments borrow money to meet exceptional expenditure, and undertake to pay a fixed rate of interest.

The promissory notes given in return for this money are called **Bonds.** The bonds differ, however, from the ordinary promissory notes in being more formal, and in having small certificates attached to enable the holder of the bond to easily collect his interest. These certificates are called **Coupons.** There is a coupon for each 3 mo. of interest. Therefore a twenty-year bond has eighty coupons attached.

Railway and other companies generally issue bonds of a nature similar to that of government bonds.

271. A person investing money in bonds is sure of a specified income, while a person investing in stocks receives only his share of the profits after all expenses, including the interest on bonds, have been paid.

272. The public debt of the United States Apr. 1, '95.

Amount of Bonds.	Rate.	When Redeemable.
$25,364,500	2%.	Option of U.S.
559,624,850	4 "	July 1, 1907.
54,710	4 "	
100,000,000	5 "	Feb. 1, 1904.
28,807,900	4 "	Feb. 1, 1925.

$713,851,960.00	Total int.-bearing debt.
381,025,096.92	Non int.-bearing debt (U.S. Notes, Nat. Bank Notes, Fractional Currency).
1,770,250.26	Debt which has matured.
$1,096,647,307.18	Total debt, exclusive of bonds issued to Pacific railroads.

273. Stocks and bonds, except those of small companies, are bought and sold at a special market, called a Stock Exchange. The agent who is employed to buy and sell for the public is called a **Stock Broker**, and the person who deals in stocks and bonds is called a **Stock Jobber**.

Stock Brokers charge for their services a commission called **Brokerage**; in calculating the cost of stocks and bonds this brokerage must be added to their market prices; the proceeds of a sale of stocks and bonds are the market prices minus the brokerage. In previous examples, brokerage has been allowed for in the prices.

274. Brokerage is generally $\frac{1}{8}$ of 1%, reckoned on the *par value* of the stock; it is therefore $\frac{1}{8}$ of \$1 on every \$100 share bought or sold, no matter what the market price.

(In the following examples each share is to be considered as \$100 par value, and $\frac{1}{8}$% is to be allowed for brokerage.)

Ex. *A man sold out* \$5000 *stock of a company which paid* $3\frac{1}{2}$% *annual dividends at* $94\frac{1}{4}$, *and invested the proceeds in a stock which paid* 4% *at* $108\frac{1}{8}$; *what was his change in income?*

\$5000	$\times .03\frac{1}{2}$	= \$175	= original income.
$91\frac{1}{4}$	$-\frac{1}{8}$	= $\$91\frac{1}{8}$	= proceeds from one share.
$\$91\frac{1}{8}$	$\times 50$	= \$4556.25	= proceeds from 50 shares.
$108\frac{1}{8}$	$+\frac{1}{8}$	= $\$101\frac{1}{4}$	= cost of each new share.
\$4556.25	÷ 101.25	= 45	= number of new shares.
\$4500	$\times .04$	= \$180	= income from new shares.

∴ he had an increase of \$5 in his income.

EXAMPLES CX.

1. What is the difference between a dollar of stock and a dollar's worth of stock?

2. What is the difference in the interests on a hundred-dollar stock and a hundred-dollar bond?

3. What amount of bonds at $97\frac{3}{8}$ can be bought for \$3900?

4. What amount of bonds at $96\frac{7}{8}$ can be bought for \$5335?

5. What number of bonds at $97\frac{7}{8}$ can be bought for \$7154?

6. What number of bonds at $97\frac{1}{4}$ can be bought for \$584.25?

How much would be realized by selling

7. \$1000 bonds at 96?

8. \$500 bonds at $98\frac{1}{4}$?

9. \$100 bonds at $118\frac{1}{8}$?

10. Bonds bought at $124\frac{7}{8}$ pay 5% on the investment; what rate do they bear?

11. Bonds bought at $92\frac{7}{8}$ pay $4\frac{78}{93}$% on the investment; what rate do they bear?

12. What is the price of U.S. 5 per cents when the investment produces $4\frac{6}{11}$%?

13. I have \$10000 to invest in U.S. 4's at $118\frac{7}{8}$; what is my income, and how much money is not invested?

14. I have \$7000 to invest in U.S. 2's at $107\frac{3}{8}$; what is my income, and how much money remains uninvested?

15. U.S. 2's are bought at $114\frac{1}{8}$; what rate do they bear?

16. The trustees of a school invested, as a teachers' fund, \$40512.50 in U.S. 5's at $115\frac{5}{8}$; the salary of the principal was \$1000; how much was left for his assistant?

17. A speculator invested in a company and received a dividend of 6%, which was $8\frac{1}{3}$% on the investment; at what price did he purchase?

18. A young man receiving a legacy of \$48000 invested one half in 5% railway bonds at $95\frac{7}{8}$, and the other half in 6% stock at $119\frac{7}{8}$; what income did he secure?

19. A owns a farm which rents for \$320.40 per yr. If he should sell the farm for \$8010 and invest the proceeds in U.S. 4's at $111\frac{1}{8}$, will his yearly income be increased or diminished, and how much?

20. A capitalist drew the quarterly interest on his U.S. 4's, amounting to \$540, and afterwards sold the bonds at $\$124\frac{5}{8}$; what were the proceeds of the sale?

21. A lady invested \$20948.75 as follows: \$6160 in Maryland 6's at $96\frac{1}{8}$, \$8225 in manufacturing stock at $87\frac{3}{8}$ paying 8% annual dividends, and the remainder in steamboat stock at $73\frac{5}{8}$ paying 10% annual dividends; what was her total income?

English government bonds are called **Consols.**

22. A man had £2400 in the $2\frac{3}{4}$% consols; he sold out at $99\frac{1}{8}$ and invested the *proceeds* in 4% railway bonds, thereby increasing his income by £6 a yr; at what price did he buy the bonds?

23. A man having an income of £352 a yr. in the $2\frac{3}{4}$% consols, sells out at 97 and invests the *proceeds* in 4% railway bonds, thereby increasing his income £48 a yr; at what price were the bonds purchased?

CHAPTER XIV.

PROGRESSIONS.

275. A series of numbers which increases or decreases regularly is called a **Progression.**

For instance, 3, 5, 7, 9, 11, or 23, 20, 17, 14, 11, 8,
or 3, 6, 12, 24, or 81, 27, 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$,
are progressions.

It will be noticed that in the first two progressions the series are made by successive additions or subtractions, while in the last two the series are made by successive multiplications or divisions.

The first are called **Arithmetical Progressions** (increasing or decreasing).

The second are called **Geometrical Progressions** (increasing or decreasing).

ARITHMETICAL PROGRESSIONS.

276. There are five things to be considered:

the first term,	denoted by a,
the last term,	" " l,
the number of terms,	" " n,
the common difference,	" " d,
and the sum of the terms,	" " s.

277. Any three of these five being given, the other two may be found.

In the arithmetical progression,

$$7,\ 10,\ 13,\ 16,\ 19,\ 22,\ 25,$$

it is evident that the last term is a plus six d, or that the first term is l minus six d.

$$\therefore\ l = a + (n-1)\,d,$$

and
$$a = l - (n-1)\,d.$$

It is also evident that if a and l be added and the sum $\div$ 2, the result will be the middle term; and that if each term be changed so as to contain as many units as the middle term the sum of the new series will be the same as the sum of the original series.

$$\therefore\ s = \frac{a+l}{2} \times n.$$

By these formulas all examples in arithmetical progression may be solved.

Ex. 1. $a = 3$, $d = 5$, $n = 12$; *find* l *and* s.

Now $l = a + (n-1)\,d$
$= 3 + 11 \times 5$
$= 58.$

$$s = \frac{a+l}{2} \times n = \frac{3+58}{2} \times 12 = 30.5 \times 12 = 366.$$

Ex. 2. $a = 5$, $l = 17$, $n = 7$; *find* d.

Now
$$l = a + (n-1)\,d;$$
$$\therefore\ 17 = 5 + 6\,d;$$
whence
$$6\,d = 12,$$
and
$$d = 2.$$

Ex. 3. *Find* n *when* $a = 2$, $l = 30$, *and* $d = 7$.

Now
$$l = a + (n-1)\,d;$$
$$\therefore\ 30 = 2 + (n-1)\,7;$$
whence
$$7\,(n-1) = 28,$$
and
$$n - 1 = 4;$$
i.e.,
$$n = 5.$$

EXAMPLES CXI.

Written Exercises.

Answer the indicated questions.

	1.	2.	3.	4.	5.	6.
$a =$	12.	5.	1.	?	?	.24.
$l =$	?	41.	4.5.	$35\frac{1}{3}$.	18.	?
$d =$	5.	4.	?	$3\frac{2}{3}$.	3.	1.2.
$n =$	8.	?	8.	6.	6.	7.
$s =$		?	?	?		?

7. Insert 3 means between 2 and 12.

8. Find the series of 8 terms when the 3d term is 14 and the 7th term is 26.

9. Find the series of 9 terms when $a = 10.8$ and the 6th term $= 4.8$.

10. Find $2 + 5 + 8 + 11 + \cdots$ to 37 terms.

11. Find $8 + 7.75 + 7.5 + \cdots$ to 11 terms.

GEOMETRICAL PROGRESSIONS.

278. There are five things to be considered:

the first term,	denoted by a,
the last term,	" " l,
the number of terms,	" " n,
the ratio,	" " r,
the sum of the terms,	" " s.

(The ratio is the relation existing between any two successive terms. It is the constant multiplier by which any term is found from the preceding term.)

Any three of these five being given, the other two may be found.

In the geometrical progression,

$$2,\ 6,\ 18,\ 54,\ 162,$$

it is evident that the last term is a times the product of r by itself four times, *i.e.*, $a \times r^4$.

$$\left.\begin{aligned} \therefore\quad l &= a \times r^{n-1}, \\ \text{and } a &= l \div r^{n-1}. \end{aligned}\right\} \text{formula 1.}$$

It is also evident that

$$s = 2 + 6 + 18 + 54 + 162; \qquad (1)$$

multiplying the equation by the ratio,

$$3s = 6 + 18 + 54 + 162 + 486; \qquad (2)$$

subtracting (1) from (2), we have

$$3s - s = 486 - 2,$$

or
$$s(3-1) = 486 - 2;$$

whence
$$s = \frac{486-2}{3-1}.$$

Now
$$486 = rl,\ 2 = a, \text{ and } 3 = r;$$

$$\therefore\ s = \frac{rl-a}{r-1}. \quad \text{formula 2.}$$

By means of these two formulas all examples in geometrical progression may be solved.

Ex. 1. $a = 3,\ r = 2,\ n = 5$; *find* l *and* s.

Now
$$\begin{aligned} l &= ar^{n-1} \\ &= 3 \times 2^4 \\ &= 48. \end{aligned}$$

$$\begin{aligned} s &= \frac{rl-a}{r-1} \\ &= \frac{2 \times 48 - 3}{2-1} \\ &= 93. \end{aligned}$$

Ex. 2. $a = 3,\ l = 81,\ n = 4$; *find* d.

Now
$$l = ar^{n-1};$$

whence
$$81 = 3 \times r^3:$$

whence
$$r^3 = 27;$$

whence
$$r = 3.$$

Ex. 3. *Find n when $a = 3$, $l = 375$, and $r = 5$.*

Now $$l = ar^{n-1};$$
whence $$375 = 3 \times 5^{n-1};$$
whence $$125 = 5^{n-1};$$
whence $$n - 1 = 3;$$
whence $$n = 4.$$

Formula 2 becomes $s = \dfrac{a - rl}{1 - r}$ if (2) is subtracted from (1). This should be used in case of a decreasing geometrical progression.

EXAMPLES CXII.

Written Exercises.

Answer the indicated questions.

	1.	2.	3.	4.	5.	6.
$a =$	2.	11.	$\frac{1}{2}$.	?	?	1.3.
$l =$	?	352.	$\frac{625}{162}$.	$\frac{2744}{216}$.	608.	?
$r =$	5.	2.	?	$\frac{7}{6}$.	2.	1.2.
$n =$	5.	?	5.	4.	6.	4.
$s =$		?	?	?		?

7. Insert 3 geometrical means between 4 and 2500.

8. Find the series of 8 terms when the 3d term is 10.8 and the 7th is 874.8.

9. Find the series of 6 terms when $a = \frac{2}{11}$ and the fourth term is $\frac{1250}{3773}$.

10. Find $2\frac{1}{5} + 6\frac{3}{5} + 19\frac{4}{5} + \cdots$ to 10 terms.

11. Find the series of 5 terms when $a = 36.015$ and the 3d term is .735.

12. Find $28.8 + 14.4 + 7.2 + \cdots$ to 7 terms.

279. When a decreasing geometrical series is extended to a large number of terms, the last term will be so small that it will have no appreciable value.

Thus, if we continue $\frac{3}{5}, \frac{3}{25}, \frac{3}{125}, \frac{3}{625}, \frac{3}{3125}, \frac{3}{15625}$, indefinitely, the last term will be almost zero; $\therefore$ in the formula $s = \dfrac{a - lr}{1 - r}$ the lr of the numerator may be omitted, and the formula will become $s = \dfrac{a}{1-r}$, by which we may find the sum of the terms of a decreasing infinite series.

Ex. 1. *Find* $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \cdots$ *to infinity.*

$$s = \frac{a}{1-r} = \frac{\frac{1}{3}}{1 - \frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

Ex. 2. *Find the value of* $.4\dot{6}$.

Now $.4\dot{6} = .4 + .06 + .006 + .0006$, etc.

$\therefore$ the value must equal $.4 +$ the geometrical progression, .06, .006, .0006, etc.

$$s = \frac{a}{1-r} = \frac{.06}{.9} = \frac{1}{15}.$$

$$\therefore\ .4\dot{6} = .4 + \tfrac{1}{15} = \tfrac{12}{30} + \tfrac{2}{30} = \tfrac{7}{15}.$$

EXAMPLES CXIII.

Written Exercises.

1. Find $\frac{1}{5} + \frac{1}{10} + \cdots$ to infinity.
2. Find $\frac{2}{3} + \frac{4}{9} + \cdots$ to infinity.
3. Find the value of $1.41\dot{6}$.
4. Find the value of $1.\dot{5}3\dot{1}$.
5. Find the value of $3.3\dot{3}6\dot{0}$.

CHAPTER XV.

CUBE ROOT.

280. The cubes of the first 10 whole numbers should be known: they are

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000.

An integer (or a fraction) which is the cube of another integer (or fraction) is called a **Perfect Cube.**

Thus, 64 and $\frac{125}{343}$ are perfect cubes; namely, the cubes of 4 and $\frac{5}{7}$ respectively.

281. In simple cases the cube root of a number can be found by separating it into factors, as in Art. 80.

For example, to find $\sqrt[3]{9261}$.

$$9261 = 9 \times 1029 = 27 \times 343 = 3^3 \times 7^3 = (3 \times 7)^3;$$

hence, $$\sqrt[3]{9261} = \sqrt[3]{(3 \times 7)^3} = 3 \times 7 = 21.$$

EXAMPLES CXIV.

Find the cube root of each of the following numbers:

1. 10648.
2. 3375.
3. 35937.
4. 13824.
5. 19683.
6. 42875.

Find the least number by which each of the following numbers must be multiplied in order that the result may be a perfect cube.

7. 108.
8. 392.
9. 336.
10. 441.
11. 4032.
12. 7056.

282. Since,

$$10^3 = 1000,\ 100^3 = 1000000,\ 1000^3 = 1000000000,$$

and so on, it follows that

if a number has		1 digit,	its cube has either		1, 2, or 3	digits
"	"	2 digits,	"	"	4, 5, or 6	"
"	"	3 "	"	"	7, 8, or 9	"

Hence, if we mark off the digits of a given number, beginning at the units' digit, in periods of three, the last of the periods containing one, two, or three digits; then *the number of these periods will be equal to the number of digits in the cube root of the given number.*

For example, by pointing off the numbers, 2744, 32.768, 3511808, as follows, namely, 2′744, 32′.768, and 3′511′808, we see that the cube roots of these numbers contain, respectively, 2, 2, and 3 figures.

Find $(60+3)^3$.

By Art. 86, $(60+3)^2 = 60^2 + 2(60 \times 3) + 3^2$

Multiplying by $60 + 3$

$$\begin{array}{l} 60^3 + 2(60^2 \times 3) + 60 \times 3^2 \\ \qquad\qquad 60^2 \times 3 + 2(60 \times 3^2) + 3^3 \\ \hline \end{array}$$

and $(60+3)^3 = 60^3 + 3(60^2 \times 3) + 3(60 \times 3^2) + 3^3.$

The cube of the sum of any two other numbers can be expressed in a similar form.

Hence, *the cube of the sum of any two numbers is equal to the cube of the first plus three times the square of the first multiplied by the second plus three times the first multiplied by the square of the second plus the cube of the second.*

The above Theorem will enable us to find the Cube Root of any number.

283. To find the Cube Root of any number. The method will be seen from the following examples:

Ex. 1. *To find the cube root of* 157464.

By pointing off the figures into periods of three [Art. 282], we see that there are *two* figures in the required root.

The first figure of the root is 5, since 157000 is between 50^3 and 60^3. Subtract 50^3 from the given number, and the remainder will be 32464.

Now this remainder must consist of $3 \times 50^2 \times$ units' digit $+ 3 \times 50 \times$ sq. of units' digit + cube of units' digit, and the first of these three terms is the largest; therefore if we use 3×50^2 as a trial divisor, we obtain a quotient, namely 4, which is equal to, or greater than, the unknown (units') digit. If now we add to the

$$
\begin{array}{rr|l}
 & 50^3 = & 157'464(50+4 \\
 & & 125\,000 \\
\hline
3 \times 50^2 & = 7500 & 32\,464 \\
3 \times 50 \times 4 & = 600 & \\
4^2 & = 16 & \\
\hline
\multicolumn{2}{l|}{3 \times 50^2 + 3 \times 50 \times 4 + 4^2} & 32\,464
\end{array}
$$

trial divisor the last two of the above three terms (omitting the units' digit once as a factor), we shall have as a true divisor $3 \times 50^2 + 3 \times 50 \times 4 + 4^2 = 8116$. Multiplying this by the units' digit and subtracting the product from 32464, we have no remainder.

Ex. 2. *Find the cube root of* 13312053.

237

$$
\begin{array}{rr|l}
 & 200^3 = & 13'312'053(200+30+7 \\
 & & 8\,000\,000 \\
\hline
3 \times 200^2 = & 120000 & 5\,312\,053 \\
3 \times 200 \times 30 = & 18000 & \\
30^2 = & 900 & \\
\hline
 & 138900 & 4\,167\,000 \\
\hline
3 \times 230^2 = & 158700 & 1\,145\,053 \\
3 \times 230 \times 7 = & 4830 & \\
7^2 = & 49 & \\
\hline
 & 163579 & 1\,145\,053
\end{array}
$$

Here there are three periods, and therefore three figures in the root; and, since 13000000 lies between 200^3 and 300^3, the first figure of the root is 2. Subtract 200^3, and the remainder is 5312053. Now take 3×200^2, that is 120000, as a 'trial divisor'; and 5312053 ÷

120000 will give 40 for quotient. It will, however, be found on trial that 40 is too great, for

$$(3 \times 200^2 + 3 \times 200 \times 40 + 40^2) \times 40$$

is greater than the remainder 5312053; we therefore try **30.** Take

$$3 \times 200^2 + 3 \times 200 \times 30 + 30^2,$$

and multiply this sum by 30 and subtract the product from 5312053; we shall then have subtracted altogether $(200 + 30)^3$ from the given number, and the remainder will be found to be 1145053.

To find the last figure of the root use 3×230^2, that is 158700, as a 'trial divisor,' and $1145053 \div 158700$ gives 7 for quotient. Take $3 \times 230^2 + 3 \times 230 \times 7 + 7^2$, and multiply this sum by 7, and subtract the product from 1145053. There is now no remainder; and, from Art. 219, we have now subtracted altogether $(230 + 7)^3$; hence the given number $= 237^3$, so that 237 is the required cube root.

Ex. 3. *Find the cube root of* 252.435968.

The pointing must be begun from the units' figure, and carried forwards for the integral part and backwards for the decimal part.

$$\begin{array}{rr|l}
 & & 252'.435'968'(6 + .3 + .02 \\
 & 6^3 = & 216. \\
3 \times 6^2 = & 108 & 36.435968 \\
3 \times 6 \times .3 = & 5.4 & \\
(.3)^2 = & .09 & \\
\hline
 & 113.49 & 34.047 \\
\hline
3 \times (6.3)^2 = & 119.07 & 2.388968 \\
3 \times 6.3 \times .02 = & .378 & \\
(.02)^2 = & .0004 & \\
\hline
 & 119.4484 & 2.388968 \\
\end{array}$$

The process can be somewhat shortened, as in Square Root; it is, however, very rarely necessary to find a cube root, and it is therefore undesirable to attempt to shorten the above process.

EXAMPLES CXV.

Find the cube root of each of the following numbers:

1. 1331.	7. 79507000.	13. 2.197.
2. 3375.	8. 148877000.	14. .004913.
3. 4913.	9. 8869743.	15. .238328.
4. 12167.	10. 733870808.	16. 125525.735343.
5. 29791.	11. 2352637.	17. $2\frac{10}{27}$.
6. 68921.	12. 16974593.	18. $39\frac{38}{125}$.

19. $12568\frac{5}{64}$. 20. $240\frac{17712}{19683}$.

Find to three significant figures:

21. $\sqrt[3]{10}$. 22. $\sqrt[3]{1.5}$. 23. $\sqrt[3]{3.75}$. 24. $\sqrt[3]{.0675}$.

25. Find the side of a cube which has the same volume as a beam 40 ft. 6 in. long, 1 ft. 4 in. wide, and $\frac{3}{4}$ in. thick.

26. Find the length of one edge of a cube whose volume is 2 cu. yd. 14 cu. ft. 145 cu. in.

27. Find the area of each face of a cube whose volume is 5 cu. yd. 2 cu. ft. 1592 cu. in.

28. Find approximately the length of one edge of a cubical vessel which contains a gallon.

29. Find approximately the side of a cube of iron which weighs a t., assuming that a cu. ft. of iron weighs 486 lb.

30. Find, to the nearest mm, the length of a cube of gold which weighs as much as a cu m of water, the S.G. of gold being 19.5.

MISCELLANEOUS EXAMPLES FOR GENERAL REVIEW.

1. Express in words 5006017, and in figures thirteen million twenty-five thousand eleven.

2. Find the least multiple of 3157 which is greater than a million.

3. How many articles each worth \$14.45 should be given in exchange for 60 articles each worth \$49.13?

4. Reduce 5 t. 7 cwt. 30 lb. 11 oz. to oz.

5. Find the G.C.M. also the L.C.M. of 3432 and 3575.

6. Find the sum of $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{11}{12}$, and $\frac{13}{24}$.

7. Divide $43\frac{1}{43}$ by $28\frac{28}{129}$, and express the result as a fraction of 12.

8. Divide .221312 by 5.32.

9. Add .375 of 13*s.* 4*d.* and .07 of £2. 10*s.*, and subtract the result from £.45.

10. Find the rent of 134 A. 145 sq. rd. at \$19.50 per acre.

---

11. Multiply 905741 by 518963, and express the result in words.

12. A certain number was divided by 77 by short divisions; the quotient was 137, the first remainder was 9 and the second remainder was 6; what was the dividend?

13. Reduce 15 m. 95 rd. 3 yd. to in.

14. A grocer mixed 48 lb. of tea which cost him 64 ct. a lb. with a certain quantity which cost 60 ct. a lb. He then sold the whole for \$76.92, and gained \$7.20 by the transaction. How much tea did he sell?

15. Express 756, 1155, and 1176 as the products of prime factors.

16. Simplify $4\frac{1}{2} + \frac{1}{3} - 3\frac{3}{4} + 5\frac{7}{8} - 6\frac{11}{16}$.

17. Simplify $\frac{3}{4}$ of $\frac{5}{12}$ of $4\frac{8}{13} \div 1\frac{2}{3}$ of $\frac{5}{13}$.

18. Simplify $2.9015 \times .01702 \times .005803$.

19. Find $\frac{3}{7}$ of £2. 11*s.* 11*d.* $-$ .115625 of £1 + .75 *d.*

20. A farm of $500^{\text{Ha}}\ 91^{\text{a}}$ is rented at $3.25 per Ha.; what is the whole rent?

---

21. Find the difference between seventy-six million eight, and four hundred ninety-nine thousand four hundred forty; and divide the result by ninety-nine.

22. What is the greatest number which will divide 2000 with remainder 11, and will divide 2708 with remainder 17?

23. Multiply 190 rd. 9 in. by 144.

24. Taking the average length of a lunar month from full moon to full moon to be 29.5306 da. and the length of a yr. to be 365.2422 da., show that 4131 lunar months are very nearly equal to 334 yr.

25. What is the least number of gr. which is an exact number both of lb. Troy and of lb. Avoir.? If the number of lb. Troy in a certain weight exceed the number of lb. Avoir. by 496, what is that weight in gr.?

26. Simplify $\dfrac{3\frac{1}{2} \times 3\frac{1}{2} \times 3\frac{1}{2} - 3}{3\frac{1}{2} \times 3\frac{1}{2} - 3} \div 8\frac{23}{37}$.

27. Find the value of a property if the owner of $\frac{3}{7}$ of it can sell $\frac{4}{5}$ of his share for $492.

28. Divide .00625 by 2500, and 6.25 by .0025.

29. Express 20 lb. 8 oz. 9 dwt. 6 gr. as a decimal of 254 lb. 10 oz.

30. Find the difference between the value of 13 cwt. 74 lb. of sugar at \$5 per cwt., and that of 52 lb. 12 oz. of tobacco at \$120 per cwt.

---

31. Write MDCCCXCIX in Arabic figures, and express 1489 by means of Roman numerals.

32. A man takes 100 steps a minute, and the average length of his step is 30 in.; how far will he walk in 4 hr.?

33. How much coal is required to supply 12 fires for 27 weeks, each fire consuming 1 cwt. 42 lb. of coal weekly?

34. Find the greatest number by which when 4344 and 5943 are divided the remainders will be 31 and 41 respectively.

35. Simplify $267\frac{3}{4}$ of $\frac{16}{19} \times (\frac{5}{9} - \frac{3}{7} - \frac{1}{15})$.

36. Simplify $\dfrac{4\frac{1}{5} - 3\frac{1}{4}}{14\frac{1}{5} + 4\frac{1}{4}} \div \dfrac{4\frac{1}{5} \text{ of } 4\frac{3}{4}}{13\frac{13}{14} \div 1\frac{24}{41}}$.

37. If $\frac{18}{59}$ of $1\frac{1}{3}$ of $29\frac{1}{2}$ of a certain sum is \$1692.60, what is the sum?

38. Reduce $.\dot{6}\dot{3}$, $.48\dot{3}2\dot{4}$, and $.016\dot{5}\dot{4}$ to common fractions in their lowest terms.

39. What decimal of \$2.25 is \$5? Find the value of .78125 of \$4 – .0625 of \$1.20 – 2.75 of \$.04.

40. What is the cost of a silver cup weighing 2 lb. 5 oz. 17 dwt. 12 gr. at \$1.85 per oz.?

---

41. \$603.42 is to be divided equally among 226 people; how much will each receive?

42. The heights of 5 boys are respectively 5 ft. $4\frac{1}{2}$ in., 5 ft. 2 in., 5 ft. $1\frac{1}{2}$ in., 4 ft. 10 in., and 4 ft. $8\frac{1}{2}$ in.; what is the *average* height?

43. Reduce 726314 in. to mi., rd., etc.

44. A circular running path is 902 yards round. Two men start back to back to run round, and one runs at the rate of 10 miles and the other at the rate of $10\frac{1}{2}$ miles an hour. When and where will they meet for the first time?

45. Find the greatest length of which both 42 yd. 9 in. and 55 yd. 9 in. are multiples.

46. Find the least fraction which added to the sum of $\frac{11}{56}$, $\frac{8}{49}$, and $\frac{13}{80}$ will make the result an integer.

47. What fraction of $27 is $\frac{9}{11}$ of $1.21?

48. Simplify $\dfrac{5.76}{.018} \times \dfrac{.00196}{.64}$; and divide $.\dot{7}\dot{2}$ by $.\dot{1}1793\dot{6}$, expressing the result as a recurring decimal.

49. Find the value of $.0\dot{5}$ of $.\dot{1}0\dot{1}$ of £74. 18*s.* 6*d.*

50. A person buys 5 cwt. 46 lb. of sugar at $\$3.87\frac{1}{2}$ per cwt., and sells it at 4 ct. per lb; what is the gain?

---

51. Find the sum of all the numbers between 100 and 200 which are divisible by 13.

52. If a person's income be $1700 a year, find what he will save in 4 yr. after spending on an average $25.50 a week, taking 52 weeks to a yr.

53. Divide 69 mi. 319 rd. 2 yd. 1 ft. 10 in. by 136.

54. The L.C.M. of two numbers is 11160, the G.C.M. is 15, and one of the numbers 465; what is the other number?

55. Simplify $\frac{1}{9}$ of $1\frac{1}{4}$ of $4\frac{1}{2} \div \frac{5}{36}$ of $1\frac{1}{3}$ of $3\frac{1}{2}$.

56. Express $\dfrac{1}{3+\dfrac{3}{5+\dfrac{5}{7+\frac{7}{9}}}}$ as a simple fraction.

57. Subtract $\frac{13}{14}$ of $\frac{1}{6}$ of \$21 from $\frac{2}{5}$ of $\frac{7}{16}$ of \$20; and express the difference as a fraction of an eagle.

58. Divide .37592 by .0125, and 3759.2 by .000125.

59. Express $\frac{3}{8}$ of $2.624^{\text{Hl}} - \frac{14}{5}$ of 1.375^{l} as Kl. Is the answer numerically the same as cubic meters?

60. Find the value of 7 A. 80 sq. rd. of land at \$200 per A.

---

61. How many mi., etc., are there in a hundred million in.?

62. If butter be bought at \$27 per cwt. and sold at 33 ct. per lb., how much will be gained on every cwt.?

63. Find the G.C.M. of 1035, 391, and 598.

64. Simplify $4\frac{1}{4}$ of $3\frac{1}{3} - 2\frac{1}{2} \div 5\frac{5}{7} + 6\frac{6}{11} \div 3\frac{3}{11}$.

65. Simplify $\dfrac{2}{3 - \dfrac{4}{5 - \frac{3}{2}}}$ and $\dfrac{1551.55}{65.1} \times \dfrac{21}{20.02}$.

66. A square cistern is 3^{m} long inside and when filled contains 47.25^{T} of water; what is the depth of the cistern inside?

67. If a bankrupt pays \$23 in a hundred, how much will a creditor, to whom he owes \$7866, receive?

68. If 8 cwt. 20 lb. cost \$20.50, what would a t. cost at the same rate?

69. The distance between two stations is 234 mi. 160 rd. 38 yd. 2 ft. An engine wheel revolves 142878 times in traveling from one station to the other. How many in. does it travel in one revolution of the wheel?

70. What is the greatest weight of which both 2 t. 4 cwt. 18 lb., and 5 t. 5 cwt. 94 lb. are multiples?

---

71. The sum of the ages of a father and of his son is now 88 yr., and 12 yr. ago the father was three times as old as the son; how old are they?

72. A number is divided by 210 in three steps, the factors being 5, 6, 7 in order; and the remainders are 2, 3, 4 in order; what would have been the remainders if the number had been divided by 7, 6, 5 in order?

73. Reduce 216875 in. to mi., etc., also 57637 sq. yd. to A., sq. rd., and sq. yd.

74. Find, to within a thousandth of the whole, the square roots of 15, $\frac{3}{80}$, and .081.

75. Find two numbers, one of which is double the other, and whose product is 8192.

76. Find the following:

$$17 \times 19,\ 18 \times 14,\ 13 \times 16,\ 19 \times 15,$$
$$75^2,\ 95^2,\ 105^2,\ 115^2.$$

77. Find the least length which is a multiple of 1 ft. 6 in., 4 ft. 6 in., 7 ft. 6 in., and 15 ft. 9 in.

78. What is the acreage of a rectangular field whose sides are respectively 201 yd. 2 ft., and 60 yd.?

79. A rectangular field contains 2 A. 134 sq. rd., and its length is 6.25 ch.; what is its breadth?

80. What is the least length of carpet 27 in. wide that would be required to cover the floor of a room 24 ft. long and 21 ft. wide?

---

81. Simplify $4\frac{3}{7} \div (\frac{5}{28} \times \frac{4}{15} \times 10\frac{1}{3})$ and $4\frac{3}{7} \div \frac{5}{28} \times \frac{4}{15} \times 10\frac{1}{3}$.

82. Simplify $7 \times 16 - \frac{2}{3}$ of $4\frac{1}{2} \div 1\frac{1}{2} \times 17 - \{4(18 - 6) + (26 - 3)\} - 7$.

83. Find by factors $\sqrt{1936}$, $\sqrt{2601}$, $\sqrt{\frac{225}{23716}}$.

84. How much will it cost to paint the ceiling of a room 15 ft. 6 in. long and 12 ft. 6 in. wide at 16 ct. per square foot?

85. How many loads (cu. yd.) of gravel would be required to cover to a depth of 2 in. a path 90 yd. long and 5 ft. wide?

86. One side of a square field of $22\frac{1}{2}$ A. abuts on a road. This side is divided into building plots 100 ft. deep and having a frontage along the road of 30 ft. each. The building plots are let at £12 each, and the rest of the field at £5. 10*s.* an A. What is the total rental of the property?

87. A dealer purchased 40 tubs of butter, each containing 35 lb., at 22 ct. per lb., and sold 35 tubs of the butter for as much as the whole cost; for how much per lb. must he sell the remainder in order to gain 16% and \$3.22?

88. What is the acreage of a rectangular field whose length is 117 rd. and whose breadth is 55 rd.?

89. The number of sheep on a farm increased for 4 yr. at the rate of 20% each year, and there were originally 625 sheep; how many were there at the end of the 4 yr.?

90. A man makes a profit of 20% by selling an article for 24 ct.; how much % would he make by selling it for 25 ct.?

---

91. From a vessel containing 32.3^{l} of kerosene 1722.5^{cl} were drawn; how many dl remained?

92. Find the least number which when divided by 17 leaves a remainder 12, and when divided by 29 leaves a remainder 24.

93. Reduce to their simplest forms:

(i) $\frac{3}{11}(\frac{2}{3}+\frac{5}{6}+\frac{7}{12}-\frac{1}{4})-1\frac{1}{4}$ of $\dfrac{1}{3\frac{1}{3}}$.

(ii) $(\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5})\div(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5})$.

94. The age of a father is three times the sum of the ages of his three sons, and two years ago the father's age exceeded the sum of the ages of the three sons by 36 years; how old is the father?

95. A body weighs 60^g in air and 42^g in water; what is its S.G.?

96. Find the interest on a 30 da. Mass. note for $7895.56.

97. A man bought $13\frac{7}{8}$ bu. of corn for $7.77, and sold the same at 20% profit; what was the selling price per bu.?

98. A man paid $45.10, including a duty of 10%, for a watch; how much was the duty?

99. The distance between two places on a map is 156^{mm}; what is the distance in Km if the scale of the map is 1 to 80000?

100. Find the number of Km in one mi.

---

101. Find the prime factors of the L.C.M. of 391 and 493.

102. If 15% be lost by selling an estate for $3400, for what must it be sold to gain 20%?

103. Find $\sqrt{.6}$ to the nearest thousandth.

104. A beam 36 ft. long, and whose section is a square, contains $182\frac{1}{4}$ cu. ft. of timber; what is its width?

105. Find the length of the side of a square field which contains 10 A.

106. Divide 570326 by 63 by 'short' divisions, explaining clearly the formation of the remainder.

107. Reduce to its simplest form

$$\left(\tfrac{1}{2}+\tfrac{2}{3}\right) \text{ of } \left(\tfrac{3}{4}+\tfrac{4}{5}\right)+\tfrac{5}{6} \text{ of } \left(\tfrac{1}{8}+\tfrac{1}{10}\right)+\frac{1\frac{1}{2}}{22\frac{1}{2}} \div \frac{21\frac{1}{3}}{1\frac{1}{3}}.$$

108. Multiply 36.2 by .057, and divide 5752.8 by .00376, and .0025 by 3.1.

109. Find the value of a bar of gold weighing 5 lb. 10 oz. 17 dwt. 22 gr. at $20 per oz.

110. How many gallons will a cistern 6 ft. by 4 ft. by 3 ft. hold?

---

111. The total number of votes given for two candidates at an election was 127345, and the successful candidate had a majority over the other of 17377; how many votes did each get?

112. Divide $875 between three persons so that the first may have $50 more than the second, and the second $75 less than the third.

113. A certain number less than 1000, when divided by 56 or by 72 leaves 13 as remainder; what is the number?

114. A grocer mixes 9 lb. of coffee at 54 ct. a lb. with 6 lb. of chicory at 15 ct. a lb; at what price per lb. must he sell the mixture in order to get a profit of 25%?

115. The breadth of a room is twice its height and the length is thrice its height; and it cost $115.20 to paint the walls at $.08 per sq. ft.; what is the height?

116. How many turfs each 3 ft. by 1 ft. would be required to turf a lawn 96 ft. by 75 ft., and how much would they cost at $1.75 a hundred?

117. Find the weight of a rectangular solid piece of iron 17^{cm} by 5^{cm} by 3^{cm}, the S.G. of iron being 7.8. Answer in Kg.

118. Find the interest on $672.87 for 2 yr. 7 mo. at 4%.

119. In a room 22 ft. by 18 ft. there is a Turkey carpet with a border 2 ft. wide all round it. The carpet cost 20 *ga.*; how much was that a sq. yd.

120. Find the length of a square field whose area is 4 A. 89 sq. rd.

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121. A wire .2346 yd. long is cut up into pieces each .007 yd. long; how many pieces will there be, and what length will be left over?

122. A room is 21 ft. long, 17 ft. wide, and 12 ft. high; how many pieces of paper 21 in. wide and 12 yd. long must be bought to paper the room supposing 150 sq. ft. of the walls are left uncovered?

123. A class contains 19 boys; and in an examination 6 boys got 56% of the full marks each, one got 90%, and the rest got 39% each, except one boy who got no marks at all; what was the average % got by the boys in the class?

124. A rectangular block of timber is 5 ft. long and contains 3 cu. ft. If its section be a square, find its thickness to the nearest tenth of an in.

125. A square field is bordered by a path one yd. wide, the field and path together occupying two and one half A.; find the cost of covering the path with gravel at 36 ct. per sq. yd.

126. A flask holding 25^{ccm} of water, holds 20.25^{g} of alcohol; find the S.G. of the alcohol.

127. Some goods cost $25; how much is lost by selling them at 20% below cost?

128. One lb. Troy is what % of one lb. Avoir.?

129. What is the proceeds of a N. Y. note for $2040 drawn Jan. 31, '95, at 3 mo. and discounted on Feb. 25th at 5%?

130. What sum is invested if the investment yields $585 per annum at $4\frac{1}{2}$%?

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131. Reduce 563147 in. to mi., etc.

132. Find the prime factors of 58212. What is the greatest square number of which 58212 is a multiple?

133. Simplify $\dfrac{2\frac{109}{180} + \frac{1}{4}\left(\frac{2}{3} + \frac{3}{5}\right) - \frac{2}{3}\left(\frac{1}{4} + \frac{3}{5}\right)}{\frac{3}{4} \text{ of } \frac{5}{6} - \frac{2}{3} \text{ of } \frac{4}{7}}$.

134. Find the acreage of a rectangular field whose length is 25 ch. 80 li. and whose breadth is 8 ch. 75 li.; find also the rent at $12 an A.

135. A certain piece of work can be done by 8 men or 16 boys in 10 da. In how many da. can the work be done by 8 men and 16 boys?

136. An object weighs 10^{g} in air and 4^{g} in water; find its S.G.

137. A man, after deducting $4000 from his income, pays $170 income tax on the remainder. If the $4000 had not been deducted, the tax would have been $250. Find the rate of taxation and the income.

138. A demand note with interest was paid 4 yr. after date. The interest at $4\frac{1}{2}\%$ was $365.04; find the principal.

139. A demand note bearing interest was paid 4 yr. after date. The amount at 5% was $2433.60; find the principal.

140. A train 110 yd. long was observed to pass a certain point in 10 sec.; how many mi. an hr. was it then going?

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141. Determine the number which when divided by 231 by the method of 'short' divisions, gives a quotient 583, and 2, 6, and 10 as successive remainders.

142. Find the G.C.M. of 464321 and 683111, and hence find all the common measures of those numbers.

143. Find the weight in Kg of the air in a room 60 ft. long, 36 ft. wide, and 21 ft. high, assuming that one cu. yd. $= .765^{\text{cu m}}$, and that air weighs 1.29^{g} per liter.

144. The wages of A and B together for 45 da. amount to the same sum as the wages of A alone for 72 da.; for how many da. will this sum pay the wages of B alone?

145. A room is 20 ft. 7 in. long, 15 ft. 5 in. wide, and 11 ft. high. Find the number of pieces of paper, each 12 yd. long and 21 in. wide, which would have to be bought to paper the walls, supposing that windows, etc., which are not papered, make up one-sixth of the whole surface of the walls.

146. Ten loads (cubic yards) of gravel are spread uniformly over a path 180 ft. long and 4 ft. wide; what is the depth of the gravel?

147. A merchant borrowed $2000 from a Philadelphia bank for 30 da. at 5%; find the proceeds of the note.

148. An Ohio farmer sold some sheep for $475, and took in payment a 3 mo. interest-bearing note dated Jan. 6, '93, rate $5\frac{1}{2}\%$. On Mch. 1st the farmer had the note discounted at 5%; how much cash did he receive from the bank?

149. One pound of silver is weighed in water; how many pwt. does it lose, the S.G. being 10.5?

150. Find the weight in dg of a cylindrical stick of silver 10^{cm} long and 1^{cm} in diameter, the S.G. being 10.5.

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151. Find the least length which is a multiple of 5 yd. 1 ft. 3 in., and also of 7 yd. 2 ft. 9 in.

152. Simplify $(4\frac{1}{3}-2\frac{5}{8}\text{ of }\frac{3}{7}+2\frac{1}{5})\div\{(4\frac{1}{3}-2\frac{5}{8})\text{ of }\frac{3}{7}+2\frac{1}{5}\}$.

153. Find $\sqrt{783}$ to the nearest tenth.

154. (i) Multiply $17+19+16$ by 18.
(ii) Find mentally $(30+4)^2$.
(iii) Find mentally 85^2.

155. Find

$$[35^2 \div 7^2 \times 4^2 - \{(150 \times \tfrac{2}{3} \div 25) + 1260 \div 35\}]\ 11.$$

156. (i) A ratio is 47; find the second term when the first term is 235.

(ii) A ratio is $\frac{1}{9}$; find the first term when the second term is $\frac{3}{21}$.

(iii) Two similar rooms are respectively 8 yd. and 9 yd. long; how much paper will be required to paper the first room, compared with that which will be required for the second room?

157. Sound travels at the rate of 1090 ft. a second; how far off is a thunder-cloud when the sound follows the flash after $5\frac{1}{2}$ sec.? Answer to the nearest hundredth of a mi.

158. A father, who had three children, left his second son \$500 more than he left the third son, and his eldest son twice as much as the third. They had \$8500 between them; how much had each?

159. In a certain examination every candidate took either Latin or Mathematics, also 79.4% of the candidates took Latin and 89.6% took Mathematics. If there were 1500 candidates altogether, how many took both Latin and Mathematics?

160. For what sum must goods worth \$6370 be insured at 2% premium so that in case of loss the owner may recover the value both of the goods and the premium?

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161. Simplify $\dfrac{2\frac{3}{7} \text{ of } \frac{5}{9} - 8\frac{1}{3} \text{ of } \frac{26}{171}}{8\frac{1}{3} \text{ of } (\frac{6}{19} - \frac{2}{7}) \text{ of } \frac{59}{180}}$.

162. A bill of \$301.05 was paid with an equal number of eagles, dollar pieces, quarters, and five-cent pieces; how many coins of each kind were there?

163. A and B received respectively $\frac{3}{19}$ and $\frac{4}{17}$ of a certain sum of money, and C received the remainder. A received \$1173; how much did C receive?

164. What is the cost of a plot of building-land 242 ft. long and 21 ft. wide at \$2000 an A.?

165. At the beginning of a year the population of a town was 16400. The deaths during the year were 3% of the population at the beginning of the year, and 80% of the births. What was the population at the end of the year, neglecting changes caused by traveling?

166. A liter flask was half filled with sand, and the weight of the sand was 1375^{g}; what was the S.G. of the sand?

167. What is the cost of concreting the bottom of a circular pond 70 ft. in diameter, when concreting costs \$1.87 per sq. yd.?

168. Find the exact interest on \$700 for 30 da., at 6%.

169. A merchant sold goods for \$5650, with 20% and 5% discount, and 10% off for cash. Cash was paid; how much did the merchant receive for his goods?

170. The buyer of the goods in Ex. **169** sold the goods for \$5603.38; what was his % profit? What was his percentage profit?

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171. Express 1887 by means of Roman numerals.

172. Express in t. and fractions of a t. the weight of lead required to cover a flat roof, 147 sq. yd. in extent, with sheet lead one-eighth of an in. thick, supposing that a cu. ft. of lead weighs 820 lb.

173. Simplify

$$\tfrac{11}{221}\left\{\left(\frac{4\frac{1}{11}}{20}+\frac{70}{8\frac{3}{4}}\right)-1\tfrac{11}{25}\left(\frac{3\frac{2}{3}+6\frac{1}{10}}{4\frac{1}{9}-3\frac{2}{3}}\right)\div 4\tfrac{79}{200}\right\}.$$

174. Find the value of a silver cup weighing 2 lb. 7 oz. 7 dwt. 12 gr. at \$1.20 an oz.

175. Find the cost of painting the sides and bottom of a cistern 3 yd. long, 5 ft. wide, and $3\frac{1}{2}$ ft. deep at 3*s.* 9*d.* per sq. yd.

176. Two similar boxes hold 125 lb. of sand and 216 lb. of sand respectively; the larger box is 36 in. long; find the length of the smaller box.

177. Find the bank discount on a note for \$1460 payable in San Francisco 30 da. after date.

178. Find the trade discount on a bill of goods for \$1460 with 15% and 7% off.

179. The volume of a room is 2592 cu. ft.; what is the length of the room when the height is 9 ft. and the breadth is 16 ft.?

180. For economy which way would carpet strips run in the room of Ex. **179**?

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181. What number is the same multiple of 354 that 86445 is of 765?

182. Subtract $\frac{2}{3}$ of $\frac{5}{6}$ from $1\frac{1}{2}$ of $\frac{4}{9}$; and divide the result by $(\frac{2}{3}-\frac{4}{7})\times(\frac{4}{5}-\frac{5}{8})$.

183. A lidless cistern 10 ft. 6 in. long, 7 ft. 4 in. broad, and 5 ft. 4 in. high is to be painted outside; find the cost at $4\frac{1}{2}$ ct. per sq. ft.

184. A promissory note, written for 30 da. and payable in Ohio at 4%, amounts to \$1505.50; find the principal.

185. A promissory note, written for 45 da. and payable in Ohio at 4%, amounts to \$1960.40; what would have been the amount if the note had been payable in N.Y.?

186. When railroad 4's can be bought at $101\frac{1}{2}$ (brokerage $\frac{1}{8}$), how many such bonds can be bought for \$7317?

187. A man buys \$5000 of Government 4's at $111\frac{7}{8}$ (brokerage $\frac{1}{8}$); what % is he receiving on his investment?

188. A traveler purchases £500 at $4.88\frac{3}{4}$ (commission $\frac{1}{10}$); how many dollars does he pay?

189. A train moves 6 in. the 1st sec., 1 ft. the 2d sec., and so on for 75 sec., and then moves $37\frac{1}{2}$ ft. per sec. for 1 h.; how far does the train go in the 1 h. 1 min. and 15 sec.? (Ans. to the nearest thousandth of a mile.) The answer lacks how many in. of the exact result?

190. In a decreasing arithmetical progression $a = 12$, $d = \frac{1}{6}$, $n = 50$; find l and s.

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191. The nearest of the fixed stars is roughly twenty trillion mi. distant. Show that it would take light $3\frac{1}{3}$ yr. to traverse this distance at the rate of 190000 mi. a sec.

192. Simplify

$$17\tfrac{1}{2} \times \{6 - 3 \div (\tfrac{1}{2} + \tfrac{1}{3})\} - 17\tfrac{1}{2} \div \{6 - 3 \times (\tfrac{1}{2} + \tfrac{1}{3})\}.$$

193. Express 1 da. 4 hr. 31 min. $52\frac{1}{2}$ sec. as a decimal of 3 da. 4 hr. 5 min.

194. Find the value of 11 oz. 13 dwt. 8 gr. of gold at $1.02 per dwt.

195. What will it cost to carpet a room 18 ft. long and 15 ft. wide, the carpet being 27 in. wide and costing $1.05 a yd.?

196. A ship is worth $45000. For what sum must it be insured at $5 per $100 in order that the owner in case of loss may receive the value of the ship and the amount of premium paid?

197. At what rate %, simple int., will $7600 amount to $7676 in 3 mo.? (No grace.)

198. What is the price of a 4% stock, if a man who invests $4301 gets an income of $136 a year on his investment? (Brokerage $\frac{1}{8}$.)

199. A man bought $100 bonds at 89 and sold them at 95 (brokerage $\frac{1}{8}$ on each transaction) and made a profit of $86.25; how many bonds did he buy?

200. Find the mean proportional between

(i) 4 and 36;

(ii) .25 and 11^2;

(iii) .64 and 1.44.

Find a third proportional to

(iv) 2.5 and 4.5;

(v) .7 and 7;

(vi) 15^2 and 5^2.

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201. A train is traveling at the rate of 35 mi. an hr.; how many ft. does it go in a sec.?

202. Simplify $2.42 \div .0025 \times .02 \div .055$.

203. Find the rent of 375.4875 A. at $12.80 an A.

204. How many loads (cu m) of gravel will be required to cover a court-yard 20^{m} by 15^{m} to a depth of 5^{cm}, and how much will the gravel cost at 84 ct. a load?

205.

475\frac{00}{}$. Boston, Mass., Jan. 6, 1895.

Fifteen days after date I promise to pay to the order of myself

Four hundred seventy-five $\frac{00}{100}$ *Dollars*

at the Globe National Bank

Value received.

No. 205. *Due* ---- Titus Luscomb.

When was this note due? What were the proceeds?

206. How many gallons will fall on a sq. mi. in a rainfall of $\frac{1}{10}$ of an in., and how many t. will the water weigh? (1 gal. of water weighs 8.33 lb.)

207. A man has an income of £525. 5*s*. from $2\frac{3}{4}$ per cent consols. He sells out at $96\frac{1}{2}$, and buys 4 per cent Russian bonds of £100 at $95\frac{1}{2}$. What will be the change in his income? (The prices include brokerage.)

208. Find the exact interest on \$750 for 36 da. at $4\frac{1}{2}\%$.

209. Two rooms of the same height are respectively 15 ft. and 20 ft. square; what is the ratio between the numbers of rolls of paper required for the walls of the rooms? For the ceilings of the rooms?

210. The first of two similar rooms requires 94^{qm} of plastering for its walls and ceiling, and is 6^{m} long; how many qm of plastering are required for the second room, which is 7^{m} long? How many cum of mortar are required for the first room if the thickness of the plastering be 1^{cm}?

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211. Express 4 min. 12 sec. as a decimal of a week.

212. A man sold 25 articles for the same price as he paid for 35; what was his profit %?

213. The number of oz. Avoir. in a certain weight exceeds the number of oz. Troy by 17; what is the weight Avoir.?

214. If a number when divided by 391 leaves a remainder of 300, what will be the remainder when the number is divided by 17?

215. A grocer bought tea at 32 ct. per lb. and sold so as to gain 25%; the duty on tea was reduced, and he then bought and sold at 4 ct. per lb. less than before; what was his gain %?

216. A man invested \$38400 in $2\frac{3}{4}\%$ bonds at $95\frac{7}{8}$; how much stock at $109\frac{7}{8}$ could he have bought with his first semi-annual interest?

217. A man embarks his whole property in four successive ventures. In the first he gained 60%, and in each of the others he lost 20%; what was his total loss %?

218. A man spent one-third of his income on lodgings, one-fourth the remainder on food, one-fifth what was left on clothes, one-sixth of the remainder on books, and then had $1200 left; what was his income?

219. If a railroad stock pays a 7% annual dividend, at what price must the stock be bought so as to yield 4% on the investment? (Brokerage as usual.)

220. A rectangular field, whose area is 1 A. 65 sq. rd., is 137 yd. 1 ft. 6 in. long; what is its breadth?

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221. A cistern 9 ft. long, 8 ft. broad, and 6 ft. deep is supplied with water by a pump which will send in 27 gal. a min.; how long will it take to fill the cistern? (Answer to the nearest sec.)

222. A cistern 3^m long, 2.5^m broad, and 2^m deep is supplied by a pipe through which run 150^l of water per minute; how many minutes will be required to fill the cistern?

223. In the centre of a room 23 ft. square there is a carpet 18 ft. square and the rest of the floor is covered with oil-cloth which is extended 6 in. under the carpet. The carpet cost $2.25 a yd. and the oil-cloth cost 90 ct. a sq. yd.; what was the whole cost?

224. A man leaves by will $3600 to his wife, and the remainder of his property to be equally divided between his four children; and it was found that the share of each child was one-seventh of the whole property; how much did the man leave?

244. In the centre of a square court is a square of grass covering $\frac{9}{16}$ of the whole area of the court, and the side of the square of grass is 60 feet; find the cost of graveling the remainder of the court to a depth of 3 in., the gravel and labor costing \$1.08 a cu. yd.

245. A man buys eggs at 30 ct. per dozen, and sells them at \$2.80 per hundred; what is his gain %?

246. A grocer pays 24 ct., 30 ct., and 40 ct. per lb. respectively for three different kinds of tea. If he mixes weights of these teas proportional to the numbers, 6, 4, and 3, respectively, and sells the mixture at 36 ct. per lb., what profit does he make %?

247. A piece of work can be done in 48 da. by 15 men, but after 9 da. two of the men go away; in how many more da. will the men who remain finish the work?

248. By selling goods for \$45.60 a man lost 5%; what would he have gained if he had sold for \$57?

249. A dealer bought a certain number of articles at the rate of 40 in a lb., and twice the same number at the rate of 50 in a lb. He sold the whole at the rate of 36 in a lb.; how much % did he gain?

250. At \$1.12$\frac{1}{2}$ per sq. yd., it cost \$506.25 to carpet a room whose length is double its breadth, and whose height is $\frac{2}{3}$ its breadth; how high is the room?

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251. Simplify

$$\frac{65}{569}\left\{\left(\frac{3\frac{11}{13}}{25}+\frac{40}{4\frac{1}{6}}\right)-11\tfrac{1}{7}\left(\frac{4\frac{1}{9}+6\frac{3}{8}}{7\frac{5}{12}-3\frac{6}{7}}\right)\div 32\tfrac{19}{23}\right\}.$$

252. A clock is set right at noon, but when it strikes 12 that night it is 80 sec. fast; find how many minutes it will gain in a week.

253. In a race of 100 yards A can beat B by 5 yards, B can beat C by 5 yards, and C can beat D by 5 yards; how should they be handicapped for a 100 yards' race, putting A at scratch and giving him the advantage of any odd fraction of a foot?

254. What is the least number which when divided by 15 leaves a remainder 9, when divided by 35 leaves a remainder 29, and when divided by 42 leaves a remainder 36?

255. The commercial discount and interest on a certain sum for the same time and rate are $254.10 and $252 respectively; find the sum.

256. A man invests $9875 partly in a 3% stock at $104\frac{7}{8}$ and partly in a 5% stock at $152\frac{3}{8}$, and he obtained $2\frac{78}{79}$% on his outlay; how much did he invest in each stock?

257. Three persons, A, B, and C, working together complete a piece of work which it would have taken them respectively 9, 10, and 12 da. to complete if working separately. They receive in payment $25.44, which they are to divide in proportion to the quantity of work done by each; find their shares.

258. Which term of the series, 9, 12, 15, etc., is 636?

259. Find $(90+2)^2$; $(60+7)^2$; $(40+8)^2$.

260. A man had 4% Railroad Preferred Stock which brought him $664 a yr. He sold out at $119\frac{1}{8}$, and invested in Common Stock at $145\frac{1}{8}$. The Common Stock paid 6% dividends; what was his gain in income per yr.?

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261. A square field contains 22 A. 80 sq. rd.; how long will it take a boy to run round the boundary of the field at the rate of 12 mi. an hr.?

262. A man bought a certain number of eggs at the rate of one for a ct., three times the number at the rate of three for two ct., six times the number at 11 ct. per dozen, and ten times the number at the rate of 16 ct. per score, and sells them at the rate of 90 ct. per hundred, gaining by the transaction \$3.60; how many eggs did he purchase, and what did he gain %?

263. A grocer has two sorts of tea, which cost him 64 ct. and 50 ct. per lb. respectively; in what ratio must he mix them so that he may gain 25% by selling the mixture at 70 ct. per lb.?

264. A room three times as long as it is broad is carpeted at \$1.08 per sq. yd., and the walls are colored at 18 ct. per sq. yd., the respective costs being \$39.69 and \$20.16; find the dimensions of the room, making no allowance for doors, etc.

265. A borrows from B \$550 at 6%; six da. afterwards B borrows from C a certain sum at 6%; A pays his debt in 36 da.; B pays his debt in 30 da. The interest being the same in each case, what was B's debt?

266. A man who held \$24600 of $3\frac{1}{2}$'s sold out at 93 and purchased as many 4's as possible at $130\frac{7}{8}$. He sold the 4's at $139\frac{1}{8}$ and re-invested in $3\frac{1}{2}$'s at 94. What was the change in his annual income, and how much money was not re-invested?

267. Find the sum of 51 consecutive odd numbers, the greatest of which is 117.

268.

$4670^{00}. Greenfield, Mass., Dec. 29, 1891.
Three months after date I promise to pay to
the order of Chas. R. Field
Four thousand six hundred seventy $\frac{00}{100}$ Dollars
at the Franklin County Bank
Value received, with interest at $5\frac{1}{2}\%$.
No. 142. Due---- James Martin.

Discounted at 5% on Jan. 12, '92. Proceeds = ?

269.

$2500^{23}. Bangor, Me., June 6, 1894.
Twenty days after date I promise to pay to
the order of Timothy Jones
Two thousand five hundred and $\frac{23}{100}$ Dollars
at the 1st National Bank, Boston, Mass.
Value received.
No. 19. Due----- Henry Smith.

Discounted June 19 at $4\frac{1}{2}\%$. Proceeds = ?

270. Find the cost in New York of a Bill of Exchange on Paris for 4130 *f.*, when exchange is quoted at $5.16\frac{1}{4}$.

271. A person standing on a railway platform noticed that a train took 21 sec. to pass completely through the station, which was 88 yd. long, and that it was 9 sec. in passing him; how long was the train, and at what rate per hr. was it traveling?

272. Two trains start at the same time from A and B and proceed towards each other at the rate of 35 and 45 mi. per hr. respectively. When they meet, one train has gone $17\frac{1}{2}$ mi. further than the other. What is the distance from A to B?

273. A man bought a house and sold it so as to gain 5 per cent. Had he given 10 per cent more for the house, and sold it for \$129.60 more than he did sell it for, he would have lost $2\frac{1}{2}$ per cent. Find what he gave for the house.

274. An example in multiplication was worked correctly, and then all the figures except those given were erased, and the lines show the positions of the missing figures; find the missing figures.

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   4--
    3-
  ----
  36--
--7-
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--3--
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275. What is the side of a square field which contains 3 A. 96 sq. rd.?

276. A thin rectangular lamina of metal, 3 ft. 2 in. long and 2 ft. 9 in. wide, has cut from its four corners four squares whose sides are 3 in. long. The four projecting portions are turned up at right angles to the rest of the lamina, and thus form a lidless box. Find the capacity of the box.

277. Perform the last example after substituting dm for ft. and cm for in. and find out the weight of pure water the box would hold; answer in lb., etc., to the nearest gr.

278. An accommodation train going at the rate of 25 mi. an hr. starts on a journey an hour before an express train which goes at the rate of 40 mi. an hr. The accommodation train arrives 15 min. before the express. Find the length of the journey in Km.

279. On a map on the scale of 6 in. to a mi. a rectangular field is represented by a space 1 in. long and $\frac{1}{4}$ in. broad; find its area in A. Also find how many yd. less of paling would be required to enclose a square field of the same area.

280. A cubical block of metal of 7.84 in. thick weighs .25 lb. per cu. in. A hole of square sectional area is to be cut completely through the metal, perpendicular to a face of the cube, in order that the weight of metal left may be 100 lb. Find, to three places of decimals, the side of the square section.